ROBERT HOOKE'S THREE-BODY PROBLEM

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During the winter 1679, R. Hooke challenged I. Newton to predict the dynamics of an object submitted to a constant radial force. This correspondence made a strong impact on I. Newton, who wrote four years later "De Motu", the real ancestor of "The Principia", published in 1687. R. Hooke's problem can be physically linked to the dynamics of a sphere sliding on an inverted cone due to gravitational effects. If the symmetry axis of the cone is parallel to the gravitational field, the ball executes stable precessions. Breaking this symmetry induces the appearance of chaotic motions. After having derived the equations related to the position of the sphere, we analyze its dynamics, and we perform an approximated Floquet analysis that is compared to our numerical results.

Keywords: Three-body problem; inverted cone; period doubling; chaos; Floquet analysis.

1. Robert Hooke, Isaac Newton and the Inverted Cone Experiment

In spite of his essential contribution to the initial development of the science of movement, Robert Hooke is not any more known today by physicists than as the inventor of the law of Hooke which gives the back strength of a spring according to its dilation or to its contraction. Hundreds of experiments that he made in front of the members of the English Royal Society in London, the invention of numerous devices of physics, his extraordinary intuition and his work of architect in the reconstruction of London after the big fire of 1666, beside his friend Christopher Wren, give him a status comparable to that of Leonardo da Vinci [Diehl, 1952; Gal, 1996; Nauenberg, 2005a, 2005b]. It was on 23 May 1666, on the occasion of one of the numerous experimental demonstrations that he made in the "Royal Society" that R. Hooke proposed his theory of the planetary movements: "*This inflexion of a direct motion into a curve by a supervening attractive principle.*" His experiment used a pendulum constituted by a weight fixed at the end of a rope. By giving to the mass an initial velocity outside of the vertical plan formed by the rope and the vertical plumb line, he observed a pendular movement (not planar) the

projection of which on the horizontal plan described orbital movements which never passed by the center. According to the initial velocity which he gave to the mass, he reported for circular movements, elliptic movements and even movements on ellipses the main line of which turns slowly. As Galilee, he decreased the effect of the friction by using an important mass. He noticed that the acceleration of the mobile constituted by the projection of the mass on the horizontal plane is always steered towards the point upright by the mooring fixation of the rope. It is thus all about a central acceleration. We have here a mobile, the projection of the mass on the horizontal plane, subjected to an acceleration steered towards a fixed center and the movement of which is similar to that of a planet orbiting around the Sun. This experience marks the birth of "the celestial mechanics". "*The second cause of inflecting a direct motion into a curve may be from an attractive body placed in the center; whereby it continuously endeavors to attract or draw it to itself. For if such a principle to be supposed, all the phenomena of the planets seem possible to be explained by the common principle of mechanic motions; and possibly by the prosecuting [of] this speculation may give us a true hypothesis of their motions, and from some few observations, their motions may be so far brought to a certainty, that we may be able to calculate them to the greatest exactness and certainty that can be desired.*

This inflexion of a direct motion into a curve by a supervening attractive principle I shall endeavor to explicate from some experiments with a pendulous body: not that I suppose the attraction of the sun to be exactly according to the same degrees, as they are in a pendulum ..."

In other words, R. Hooke suggested composing a uniform rectilinear movement which he called "direct movement" with a movement accelerated towards a center to explain the orbital movements of the planets around the Sun [Gal, 1996; Nauenberg, 2005a, 2005b].

London, in 1670: on the occasion of a seminar in the "Royal Society", R. Hooke exposed again his theory of the orbital movement and expressed the principle of universal gravitation! "*First, thats all Cœlestial Bodies whatsoever, have an attraction or a gravitating power towards their own Centers, whereby they attract not only their own parts, and keep them from flying from them, as we may observe the Earth to do, but that they do also attract all the other Cœlestial Bodies that are within the sphere*

of their activity; and consequently that not only the Sun and Moon have influence upon the Body and motion of the Earth, and the Earth upon them..."

It is remarkable to note that we find in these texts two important ideas which we attribute generally to Isaac Newton: the Moon which "fall" permanently on the Earth by composing its uniform rectilinear inertial movement and its movement accelerated towards the center of the Earth and the universal gravitation which identifies the cause of the movements of the graves and that of celestial bodies in all the Universe.

In winter, 1679, Hooke became then secretary of the English Royal Society and he introduced a correspondence with Isaac Newton. His intention was sincerely to interest Newton in the works of the Academy. He ended however his first letter by asking to Newton for his opinion on his past works of celestial mechanics.

R. Hooke to I. Newton: Letter of November 24th, 1679:

"*From my own part I shall take it as a great favor if you shall please to communicate by Letter your objections against any hypothesis or opinion of mine. And particularly if you will let me know your thoughts of that of compounding the celestial motions of planets of a direct motion by the tangent & an attractive motion toward the central body*"

In his answer, Newton declined the offer of Hooke, pleading that he lost any interest for the questions of natural philosophy. At that time he was very busy by his alchemical activities. He however recognized not to have heard about the theory of Hooke concerning the global movements.

I. Newton to R. Hooke: Letter of November 28th, 1679:

"*...I did not before the receipt of your last letter [sent four days earlier], so much as heare ([that] I remember) of your hypotheses of compounding the celestial motion of the Planets, of a direct motion by the tan[gen]t to the curve...*"

A dialogue was established between both scholars. In his answer, Newton, proposed an experiment to demonstrate the rotation of the Earth and to this occasion, it came back to the problem of Galilee from that the movement of a body which could cross the Earth without meeting of obstacles.

As it was already mentioned, for Galilee the body released from the rest made periodic

movements to go and of return between the antipodes. Newton corrected then the proposition of Galilee by incorporating the effect of rotation of the Earth. The defenders of an immovable Earth argued that if the Earth turned from West to East, a body thrown upward would fall on the West of its initial position, there exactly where the Earth was when the body was thrown. Newton asserted on the contrary that if we release a body at the summit of a tower, because of the rotation of the Earth, it will be diverted in fact eastward. Although the description of the movement which follows is purely academic, because it corresponds to the movement of the body inside the Earth, it is particularly interesting because it informs us exactly about the understanding that Newton had in 1679 of the orbital movements. We remember that young Newton, at the age of 23, had taken refuge in his family manor house to Woolsthorpe, to flee the plague which raged in the English cities, had developed his differential and complete calculus under a very intuitive shape in direct relation with the analysis of the movement; the famous calculation of the "fluxions". He had become in some years one of the best mathematicians in Europe! He had also begun a reflection on the orbital movements, but which had stayed without continuation. It was this correspondence of the winter 79–80 with Hooke that renewed Newton's interest in these questions. The trajectory-spiral which he proposed in his letter was not correct.

The movement of the body that follows it, ends in a finite time at the center of the Earth. R. Hooke, as secretary of Royal Society, dedicated a session to the critical analysis of Newton's proposition. He recognized the abnormality to the east of trajectories, but proposed the movement of "sorts of ellipses" instead of spirals.

R. Hooke to I. Newton: Letter of December 9th, 1679:

"*. . . you seem to suppose it to descend by . . . a kind of spiral which after some few revolutions leave it in the center of the earth . . . my theory of circular motion make me suppose it would be very differing and nothing at all akin to a spiral but rather a kind of ellipsoid.*" Newton hardly appreciated to have his errors corrected. In his answer to Hooke, he made a sensational demonstration of his mathematical superiority. He knew how to calculate the trajectories of mobiles subjected to a central acceleration.
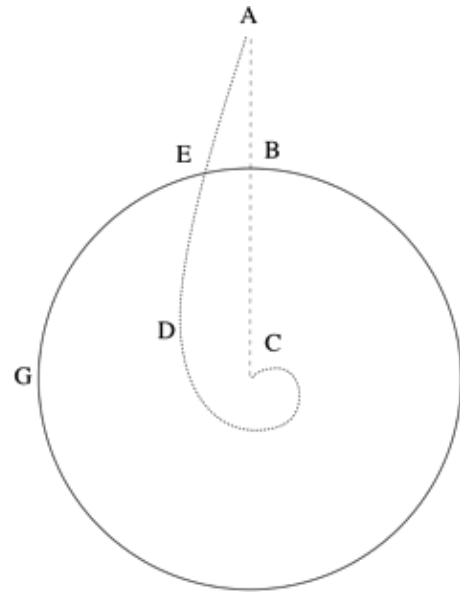

Fig. 1.   Figure drawn by I. Newton for the motion of a projectile falling inside the earth without friction.

He dealt in particular with a constant central acceleration there.

Always about the trajectory inside the Earth.

I. Newton to R. Hooke: Letter of December 13th, 1679:

"*. . . and also that if its gravity be supposed uniform, it will not descend in a spiral to the very center but circulate with an alternate ascent and descent made by its vis centriguga and gravity alternately balancing one another. Yet I imagine the body will not describe an ellipsoid but rather suit a figure as is represented . . .*"

As in 1665, Newton interpreted in an erroneous way the orbital movement as a balance between gravity and centrifugal force, but he knew how to calculate the correct trajectory by a rough method. Hooke recognized in it the movement of a ball on a concave conical surface. Once again this exchange was very instructive as regards to the state of mind of Newton. It was manifestly capable of calculating correctly the trajectory, but did not still adhere to the theory of Hooke of the composition of the inertial movement and the central attraction [Nauenberg 2005a, 2005b].

R. Hooke to I. Newton: Letter of January 6th, 1680:

"*Your calculation of the curve by a body attracted by an æquall power at all Distances from the center such that of a ball Rolling in an inverted Concave*
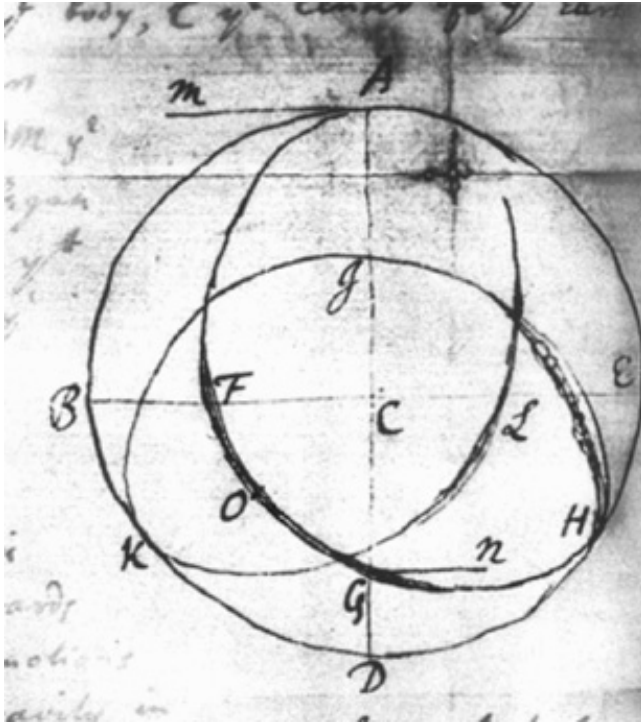
Fig. 2.   Newton diagram in his letter to Hooke on 13 December 1679.

Cone is right, and the two auges[apsides] will not unite by about a third of a Revolution.

Newton did not answer any more the letters of Hooke. This latter convinced Newton that he possessed "an excellent method" to calculate the trajectories of bodies under the influence of a central acceleration, suggesting to him to use it to discover the property of the curve produced by a central acceleration which decreased as the square of the distance in the center!

R. Hooke to I. Newton: Letter of January 17th, 1680:

"...*it now remains to know the proprieties of a curve Line (not Circular nor concentrical) made by a central attractive power which make the velocity of Descent from the tangent line or equal straight motion at all Distances in a duplicate proportion to the Distances Reciprocally taken. I doubt not but that by your excellent method you will easily find out what Curve must be, and its proprieties, and suggest a Physical Reason of this proportion*".

The correspondence between Hooke and Newton of the winter 79–80 played a considerable role in the development of the mechanics. It demonstrated with strength the power of the differential calculus that Newton invented when he was a young student in Cambridge. Indeed Newton knew how to resolve the differential equations of the movement by an approached method, what we name numerical algorithm today, without possessing the correct interpretation of these solutions.

This "collaboration" with R. Hooke led Newton four years later, in 1684, to submit to the "Royal Society" a manuscript entitled "De Motu", real ancestor of "The Principia", published in 1687. It is a work of science that is certainly the most quoted and nevertheless certainly one of the least read, since much of its reading is difficult. The "Principia" presents a revolutionary vision of the movement and unify terrestrial and celestial movements. The influence of Hooke, which Newton did not recognize, is indisputable. His idea of orbital dynamics, which composes "direct movement and central attraction" appeared in Proposition I of the book I of "Principia", an actual angular stone of the work. In this proposition, Newton demonstrated that the composition of a uniform rectilinear movement and an arbitrary central acceleration leads to an orbital movement which obeys the second law of Kepler.

2. A Sliding Ball Over an Inverted Cone as a Three-Body Problem

Motivated by these historical facts, we were interested in R. Hooke's seminal experiment of a body submitted to a constant force. We present our understanding of the dynamics of a ball sliding on an inverted cone whose axis of symmetry is tilted.

2.1. *Derivation of the equations for the dynamics*

We assume a rigid sphere located at the coordinates (X, Y, Z) that is sliding over an inverted concave cone defined as in Figs. 3 and 4:

$$Z = \tan \alpha \sqrt{\epsilon^2 + |A|^2},$$

where we use the complex notation $A = X + iY$, because of the symmetry of the surface. The parameter ϵ measures the radius of curvature near the origin, and our choice of introducing this length will become evident later in the text. Far away from the origin, the surface tends to be a cone. Following R. Hooke's experiment, we impose a tilt on the axis of symmetry of the cone. In order to keep the description as simple as possible, we then choose a gravitational field that is tilted with an angle θ, in the direction x, with respect to the z axis, as in Fig. 4.

Fig. 3. A sliding ball is moving over an inverted cone designed and constructed by Robert Magnan at *Lycée des Eucalyptus.*

In order to derive the equations of movement of the ball, we define a Lagrangian L whose associated action is minimized. We write:

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - mg(\sin\theta X + \cos\theta Z).$$

This Lagrangian was computed within the base associated to the axis of symmetry of the inverted cone. The potential energy is therefore evaluated with the rotation of an angle θ of the gravitational acceleration g. The mass of the sliding ball is m. The Euler–Lagrange equations are easily obtained:

$$\ddot{X} = -g\left(\tan\alpha\frac{X\cos\theta}{\sqrt{\epsilon^2 + |A|^2}} - \sin\theta\right) \tag{1}$$

$$\ddot{Y} = -g\left(\tan\alpha\frac{Y\cos\theta}{\sqrt{\epsilon^2 + |A|^2}}\right), \tag{2}$$
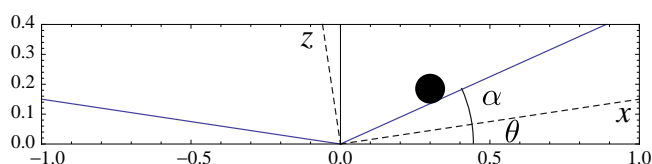

Fig. 4. A sliding ball is moving over an cone shell.

where we have neglected the dynamics in the Z direction: this is an approximation that is valid for small angles α and θ. The general treatment of the problem is done in [Argentina *et al.*, 2007], where Lagrange multiplier is introduced. This system can be further simplified by removing unnecessary parameters by using dimensionless variable. We rescale the variables X and Y with the length R, related to the apparent radius of the cone, when observed from above. We choose as a typical timescale $T = \sqrt{R/g\cos\theta\tan\alpha}$. With this rescaling procedure, we deduce the dynamics of the complex amplitude $a = A/R$:

$$\ddot{a} = -\frac{a}{\sqrt{\rho^2 + |a|^2}} - \mu, \tag{3}$$

with the two dimensionless parameters: $\rho = \epsilon/R$, and ratio of the two angles $\mu = \tan\theta/\tan\alpha$. Since we describe a mechanical system not submitted to any friction forces, the energy E is conserved:

$$E = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \mu x + \sqrt{\rho^2 + x^2 + y^2}. \tag{4}$$

This system can be studied with polar coordinates $a = re^{i\phi}$, for which we get the following set of equations:

$$\frac{d(r^2\dot{\phi})}{dt} = \mu r\sin\phi \tag{5}$$

$$\ddot{r} = -\frac{r}{\sqrt{\rho^2 + r^2}} + r\dot{\phi}^2 - \mu\cos\phi. \tag{6}$$

The first relation (5) gives the dynamics for the angular momentum $J = r^2\dot{\phi}$ in the z direction.

2.2. *Axis of the cone parallel to the gravitational field*

When the axis of the cone is aligned with the direction of the gravity, μ is equal to zero and J is a constant of the movement. As consequence, ϕ is slaved to r and the dynamics of the ball is given by the second order Eq. (6):

$$\ddot{r} = -\frac{r}{\sqrt{\rho^2 + r^2}} + \frac{J^2}{r^3}, \tag{7}$$

that describes the dynamics of a particle submitted to a potential field plotted in Fig. 5. This potential has a minimum with a nonzero value for $r = r_{\min}$, providing oscillation for r around $r_{\min}$. The oscillation in the radial direction coupled to the conservation of the angular momentum provides rosace trajectories as in Fig. 6.
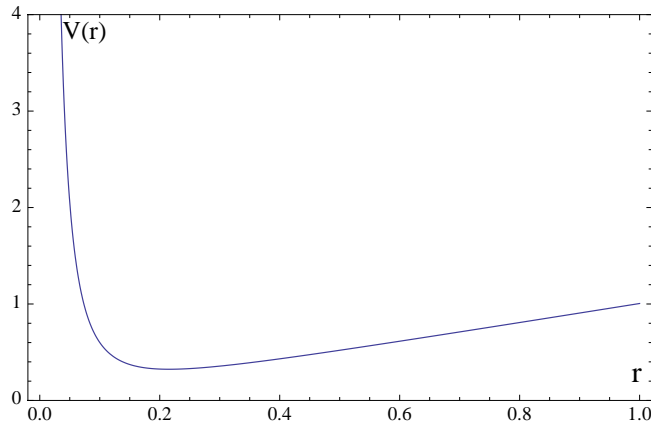
Fig. 5.   Potential field related to Eq. (7), with $\rho = 0.01$ and $J = 0.1$.

2.3.   *Singular cone*

When ρ is equal to zero the ball is sliding over a perfect cone and Eq. (3) is simplified into:

$$\ddot{a} = -\frac{a}{|a|} - \mu, \qquad (8)$$

but presents a singularity at $a = 0$. The correction parameter ρ avoids this problem. We can compute a simple solution $(x = x_0(t), y = 0)$, that represents an oscillation in the x direction, it obeys the following relation:

$$\ddot{x}_0 = -\text{sign } x_0 - \mu. \qquad (9)$$

This equation is similar to a free fall whose acceleration depends on the sign of x_0. We choose as initial
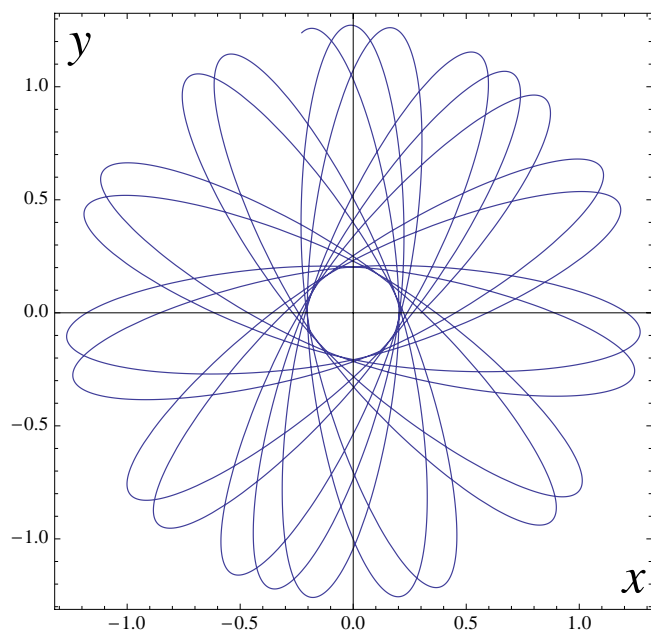

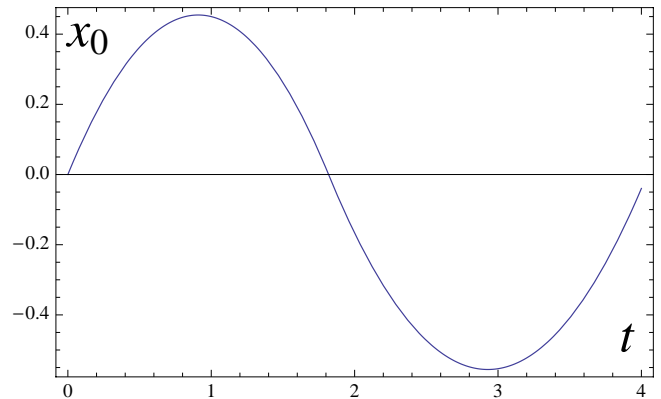Fig. 6.   Rosace trajectory obtained for $\mu = 0$ and $\rho = 0.01$.

Fig. 7.   Oscillations in the x direction for $\rho = 0$ and $\mu = 0.1$.

condition $(x_0(0) = 0, \dot{x}_0(0) = p)$. As $0 < t < \tau$, x_0 is positive, and we solve (9) with the former initial conditions, the ball is submitted to a negative acceleration. At time $t = \tau$, the ball passes in the region where $x_0 < 0$ with a velocity $-p$, the relation (9) becomes $\ddot{x}_0 = 1 - \mu$ that is solved using the initial condition $x_0(\tau) = 0$ and $\dot{x}_0(\tau) = -p$.

$$x_0(t) = -\frac{1}{2}(1 + \mu)t^2 + pt,$$

$$0 \le t < \tau = \frac{2p}{1 + \mu} \qquad (10)$$

$$x_0(t) = \frac{1}{2}(1 - \mu)t^2 - \frac{3 - \mu}{1 + \mu}pt + \frac{4p^2}{(1 + \mu)^2},$$

$$\tau \le t < T = \frac{4p}{1 - \mu^2}. \qquad (11)$$

The temporal evolution of this special solution is plotted in Fig. 7.

2.4.   *General case*

In the general case, the systems (5) and (6) represents a fourth order differential equation. Since its physical origin is mechanical, the energy
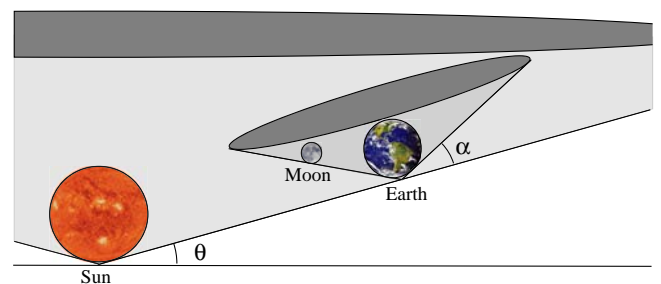

Fig. 8.   Schematic representation of the Hooke three-body problem.

is conserved during time evolution. This set of equations is then understood as a third order differential system. As μ is equal to zero, there is an additional symmetry and the angular momentum is then preserved. Consequently, with this additional conserved quantity, the dynamics is then controlled by a second order differential equation: $\dot{\phi}$ can be replaced by J/r^2 as we did in deriving (7).

As seen in Eqs. (5) and (6), the inclination of the axis of symmetry of the cone with respect to the gravitational acceleration direction removes an integral of movement: as μ is not zero, the angular momentum is not conserved any more. The resulting third order differential equation can therefore provide chaos.

R. Hooke and I. Newton worked on a description of the gravitational forces trying to understand the behavior of a ball oscillating in a cone shaped recipient. The center of the cone represents a massive object imposing its attraction to a light object: the ball. One can therefore interpret the dynamics of the ball in the cone, as the dynamics of the moon subjected to the earth's gravitational field. The inclination of the axis of the symmetry introduces an additional gravitational attraction, that can be thought as related to those of the sun. As a consequence, this very simple experiment is maybe the first model of the well known three-body problem, introduced earlier in the work of Euler in 1760.

We have performed several numerical simulations for finite value of ρ and μ, in order to explore the large phenomenology of chaotic behaviors displayed by the trajectory of a massive particle inside an inverted and inclined cone. Numerical computations are done with a fourth order Verlet algorithm [Verlet, 1967], that preserves the energy of the system on average. The energy is usually fixed to the value $E = 1$, with $\rho = 0.01$ and the typical time-step is 0.01.

For a gently tilted cone, the well-known behavior of rosace-like trajectories are still present, as shown in Fig. 6(b). But for large μ, complex behavior appears. In order to clarify where and how this complexity appear, a Poincaré section is performed at the point $x = 0$ using the Hénon Algorithm [Hénon, 1982] like in a previous study of Lopez-Ruiz and Pacheco [2002a, 2002b, 2005].

As the parameter μ is increased, islands appear on the Poincaré section denoting the presence of resonances as seen in Fig. 9(c) where homoclinic chaos is developed [José & Saletan, 1998]. The oscillation along the x direction remains marginally stable

until a critical value of μ (that depends on the regularization parameter ρ), where a period doubling bifurcation occurs as exhibited in Fig. 9(c).

2.5. *Period doubling instability*

In this section, we would like to predict the value of the parameter μ that gives rise to the period doubling instability as observed in Fig. 9(c). It corresponds to the destabilization of the vibrations with a small amplitude, along the symmetry breaking direction: $(x = x_0(t), y = 0)$, $x_0(t)$ being periodic with period T. We perform the standard linear stability analysis by studying the asymptotic behavior of perturbation of the solution. Let us define the perturbations with $x(t) = x_0(t) + \varepsilon u(t)$ together with $y(t) = \varepsilon v(t)$. Injecting this definition into (3), we get:

$$\ddot{u} = -\frac{\rho^2}{(\rho^2 + x_0(t)^2)^{3/2}}u \qquad (12)$$

$$\ddot{v} = -\frac{1}{(\rho^2 + x_0(t)^2)^{1/2}}v, \qquad (13)$$

where we have assumed that $\varepsilon \ll 1$. Since $x_0(t)$ is periodic, we need to evaluate the monodromy matrix [Joseph & Iooss, 1997], i.e. the mapping describing how a given perturbation at time t becomes at time $t + T$:

$$\begin{pmatrix} u_{i+1} \\ \dot{u}_{i+1} \end{pmatrix} = L_u(T) \begin{pmatrix} u_i \\ \dot{u}_i \end{pmatrix}, \qquad (14)$$

$$\begin{pmatrix} v_{i+1} \\ \dot{v}_{i+1} \end{pmatrix} = L_v(T) \begin{pmatrix} v_i \\ \dot{v}_i \end{pmatrix}. \qquad (15)$$

We used the notation $w_i = w(t = jT)$. The oscillation $x_0(t)$ will be stable if the norm of the perturbation decreases with time. Hence, stability will be insured if the norm of each eigenvalue (Floquet Multiplier) of the two matrices $L_u(T)$, $L_v(T)$ are smaller than one. Eigenvalues of the linear operator $L_u(T)$ are equal to one: its original system is autonomous and conservative. The appearance of the instability will therefore be related to the bifurcation in the $L_v(T)$ operator. In general, it is not possible to analytically compute the eigenvalues of $L_v(t)$, but they can be evaluated numerically; an example of the dependance of these quantities when μ is varied is shown in Fig. 10.

In fact, the solution $v(t)$ of Eq. (13) can be computed analytically, if $f(x) = \sqrt{\rho^2 + x_0^2}$ is small compared to one. In this limit, the highest derivative is multiplied by a small coefficient: it is the basis of
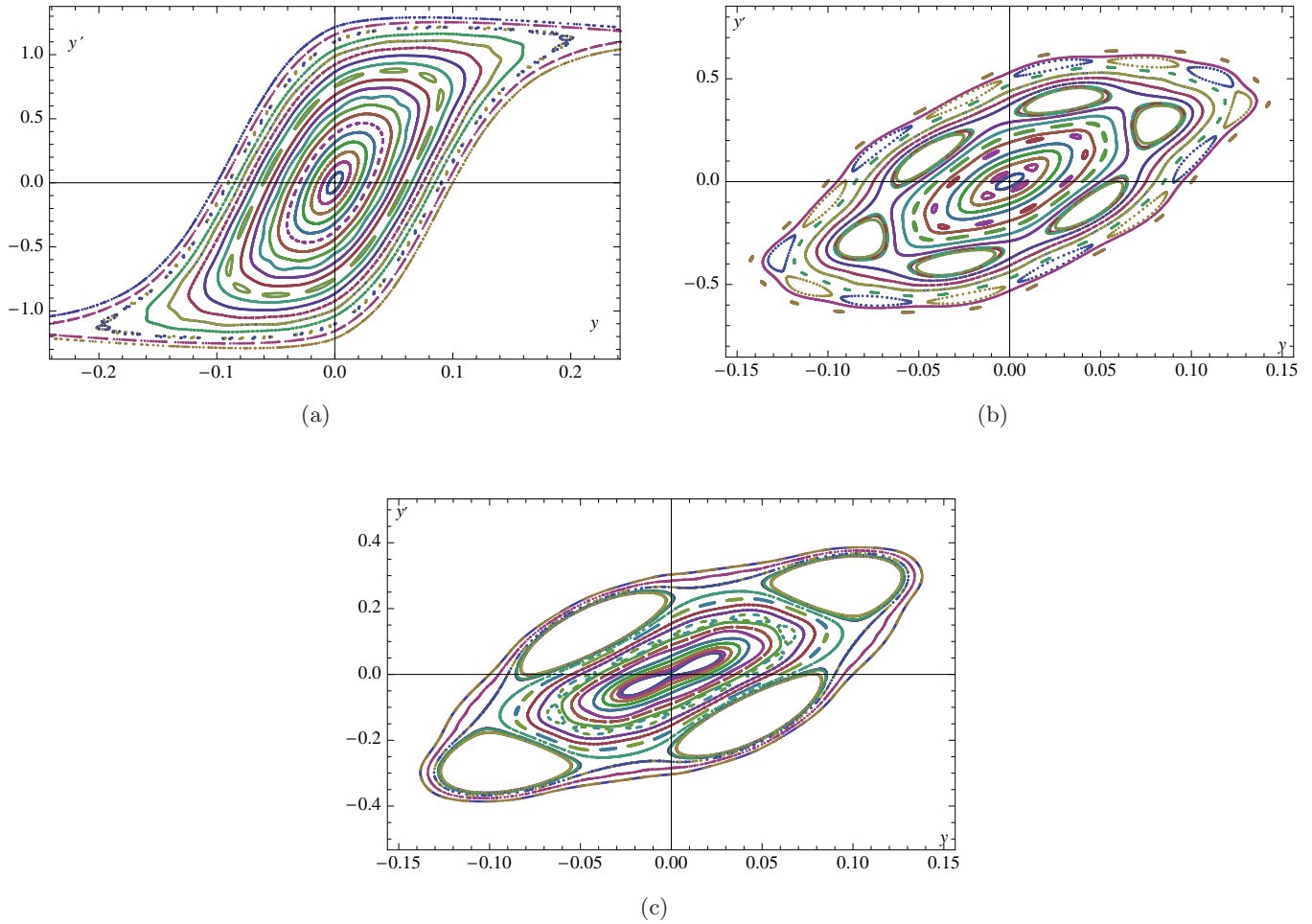
(a)

(b)

(c)

Fig. 9.   Poincaré sections taken at $x = 0$: plane $(y, \dot{y})$: (a) $\mu = 0.1$, (b) $\mu = 0.2$ and (c) $\mu = 0.3$. All these Poincaré sections have been obtained with $\rho = 0.01$ and $E = 1$.
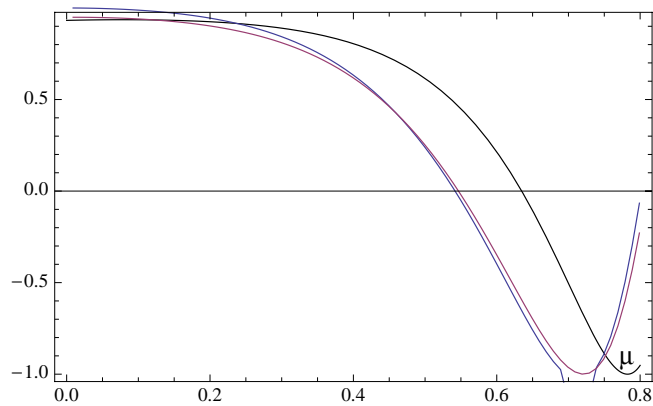

Fig. 10.   Minimum real part of the two eigenvalues of $L_v(T)$. The blue curve is computed numerically, while the red one is obtained through the WKB approximation for a numerical solution $x_0(t)$. The black curve is the analytical approximation of the Floquet multiplier using Eqs. (19) and (25). $\rho = 0.2$ and $E = 0.245$.

the well celebrated WKB approximation [Wentzel, 1926; Kramers, 1926; Brillouin, 1926]. If we write

$$f(t)\ddot{v}(t) + v(t) = 0$$

The WKB approximation gives

$$v(t) = f(t)^{1/4}(ce^{ig(t)} + de^{-ig(t)}) \qquad (16)$$

$$g(t) = \int_0^t \frac{1}{\sqrt{f(t)}} dt, \qquad (17)$$

where c and d are related to the initial condition. Equation (16) is known as the Floquet form. Taking into account this solution, we construct $L_v(T)$:

$$L_v(T) = \begin{pmatrix} \cos g(T) & \sqrt{\rho}\sin g(T) \\ -\dfrac{1}{\sqrt{\rho}}\sin g(T) & \cos g(T) \end{pmatrix}. \qquad (18)$$

We assumed that $x_0(0) = 0$, and the ball is thrown with a given and small velocity p, $\dot{x}_0(0) = p$. The Floquet multiplier $\lambda_{\pm}$ is obtained directly:

$$\lambda_{\pm} = \cos g(T) \pm i \sin g(T) \qquad (19)$$

As a consequence, the period-doubling instability occurs for $g(T) = (2i + 1)\pi$. It becomes necessary to evaluate the integral $g(T)$:

$$g(T) = \int_0^T \frac{1}{(\rho^2 + x_0^2)^{1/4}} dt, \qquad (20)$$

and this is a difficult task because we do not know precisely either x_0 nor T.

We therefore need a further approximation. We plot $x_0(t)$ in Fig. 11. It is seen, that when $x_0(t) > 0$, for t smaller than τ, $x_0 \ll \rho$. On the contrary, for $t > \tau$, $x_0(t) < 0$, and $\rho \ll x_0$. As a consequence, in order to evaluate the integral (20), we cut this later into two parts

$$g(T) = \int_0^{\tau} \frac{1}{(\rho^2 + x_a^2)^{1/4}} dt + \int_{\tau}^T \frac{1}{(\rho^2 + x_b^2)^{1/4}} dt. \qquad (21)$$

When $t < \tau$, assuming $x_0(t)$ small, the equation can be linearized, and we get:

$$x_a = \mu\rho \left(1 - \cos \frac{t}{\sqrt{\rho}} \right) + p\sqrt{\rho} \sin \frac{t}{\sqrt{\rho}}. \qquad (22)$$

It is plotted in Fig. 11. From this relation, we can compute τ, since we just need to solve $x_a(\tau) = 0$. In the limit of small ρ, we get:

$$\tau = \pi\sqrt{\rho} - 2\frac{\mu}{p}\rho + o(\rho^2). \qquad (23)$$
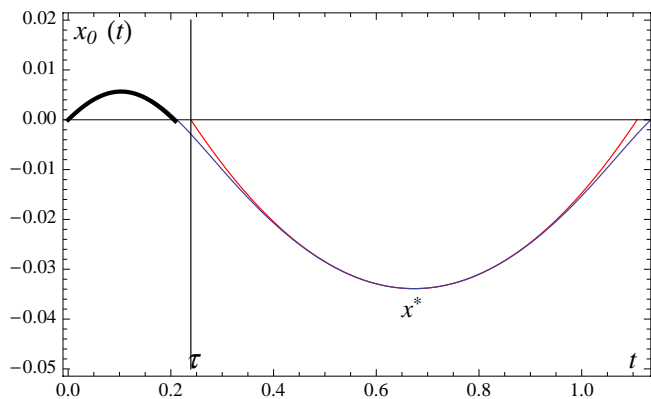

Fig. 11. Approximation of the function $x_0(t)$ represented by the blue curve. When $t < \tau$, $\rho > x_0(t)$, and we plot $x_a(t)$ in black thick line. When $t > \tau$, $x_0(t) > \rho$, and we plot $x_b(t)$ in red. $\rho = 0.01$, $p = 0.1$ and $\mu = 0.6$.
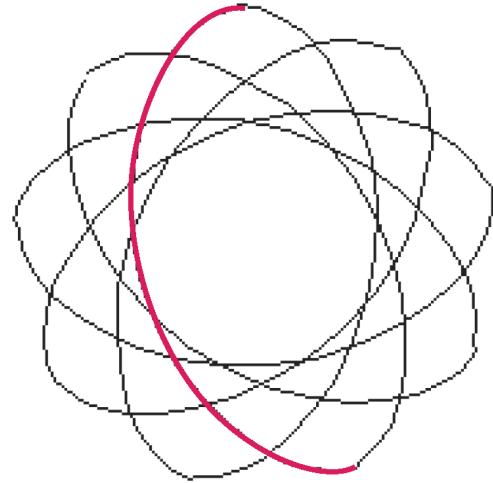

Fig. 12. Experimentally reconstructed trajectory of a ball rolling over an inverted cone. In this picture, we measured one branch of oscillation, and replicated it, mimicking the original drawing of Newton reproduced in Fig. 2.

Consequently, the first integral on the right-hand side of Eq. (21) is evaluated to $\pi - 2(\mu/p)\sqrt{\rho}$ because $x_a(t) \ll \rho$. It remains to evaluate the integral for $t > \tau$. In this region, Eq. (3) is transformed into $\ddot{x}_b + (1 + \mu) = 0$, and we can solve it:

$$x_b = x^* - \frac{1}{2}(1 + \mu)(t - t_1)^2. \qquad (24)$$

The parameter x^* is the minimum of the function x_b, and it is located at $t = t_1$. Since the energy is conserved, the minimum x^* is the root of Eq. (4) with the initial condition $(x_0(0) = 0, \dot{x}_0(0) = p)$: $p^2/2 + \rho = \mu x^* + \sqrt{\rho^2 + x^{*2}}$. The second integral on the right-hand side of Eq. (21) can also be evaluated as

$$\int_{\tau}^T \frac{1}{(\rho^2 + x_b^2)^{1/4}} dt = \frac{149}{30\sqrt{2 - 2\mu}} + o(\rho^2),$$

and we finally get the sought $g(T)$:

$$g(T) = \pi + \frac{149}{30\sqrt{2 - 2\mu}} - 2\frac{\mu}{p}\sqrt{\rho} + o(\rho^2). \qquad (25)$$

The corresponding Floquet multiplier is shown in Fig. 10.

An inverted cone in aluminium was built for an experimental demonstration. Its angle with respect to the horizontal is $\alpha = 14$. The external radius of the cone as seen from above is $130\,\text{mm}$. The radius of the internal spherical regularization seen from above is $2.5\,\text{mm}$. Hence, the dimensionless

radius is $\rho = 2.5/130 = 0.0192$. The moving ball of radius 2.2 cm is made of Latex. Its radius introduces a "cut-off" and modifies the effective nondimensional radius $\rho = 2.2/13 = 0.16$. The movements are recorded with a camera placed above the cone. Image processing allows us to follow the ball's trajectory. The resolution for the lengths is 1 mm per pixel. For null inclination, we observe a rosace-like trajectory.

In this work, we have studied the chaotic motion of a ball moving on a conical surface when the axis is slightly tilted from the direction of gravitational field. The symmetry breaking induced by the tilt is analogous to the effect of a third body in the two-body interaction. The analysis was done in the limit of a widely open cone and a small tilt.

Acknowledgments

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