Robert Hooke's conical pendulum from the modern viewpoint of amplitude equations and its optical analogues

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As stated by Lord Kelvin a long time ago, 'It seems to me that the test of "Do we or do we not understand a particular point in physics?" is, "Can we make a mechanical model of it?"'

What is the relationship between the propagation of a light wave in a Kerr medium in the presence of a magnetic field and the oscillation of a spherical pendulum on a rotating platform?

A Kerr medium is one that when submitted to an electric field its refraction index becomes a non-linear function of the latter.

It is Robert Hooke who first studied the motion of a spherical pendulum in order to approach the notion of central force. Indeed, he was willing to explore the motion of the planets with this analogous device. As a matter of fact, when a pendulum made up of a heavy mass, representing the Earth, and hanging on a wire is moved away from its equilibrium position vertically from the point of suspension, it undergoes a restoring force which tends to bring it back to the center, similar to the gravitational force exerted by the Sun on the Earth. For small amplitudes the trajectories are ellipses which precess. The ellipses are centered on the axis, in contrast to the case of planets, where the attractive center corresponds to one of the focii of the elliptical path.

Thanks to the modern formalism of nonlinear dynamics, we were able to show the close relationships between the equations which describe the motion of the pendulum and the propagation of the light wave in a Kerr medium. In both cases, an elliptical motion is induced.

It is interesting to note that the application of a magnetic field to a Kerr medium translates into an angular rotation which induces an additional precession of the pendulum—well known as the Foucault effect.

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1. Introduction

(a) Some historical facts

During the winters of 1679–1680 Robert Hooke, who had recently been appointed as General Secretary of the Royal Society, wanted to involve Isaac Newton in the activities of the academic society. In his first letter dated 26 November 1679, Hooke asked Newton to consider becoming part of life at the Royal Society and he also made him aware of his own theory of planetary motion, and in particular of the idea that one can explain the orbital paths of the planets by a combination of 'natural' (straight line) motion and of a motion subject to a centrally directed attractive force. Newton replied on 28 November, in a very polite letter, explaining on the one hand that he has lost all interest in natural philosophy and on the other hand that he had no knowledge of Hooke's orbital theory. However, he did suggest an experiment to Hooke for an attempt to demonstrate the existence of the Earth's rotation. This problem was going to be a feature of the correspondence between the two scientists for the rest of their lives; and it would eventually poison their relationship. On 13 December, in a reply to Hooke, Newton provided a drawing of an orbit of a body subjected to a constant central attraction. In his response Hooke recognized the movement as that of a ball rolling within a hollow conical surface. 'Sir, Your calculation of the path, described by a body attracted by a constant force, whatever be its distance from the centre, such as that of a ball rolling on the surface of an upturned hollow cone, is correct' Robert Hooke, in one of his numerous experiments, had indeed rolled a ball inside such a cone. And it was as a result of this experiment and of others with a conical pendulum that he acquired his prodigious insight into the subject of the motion of bodies. In this correspondence Newton commits a series of errors and these were commented on by Robert Hooke in front of members of the Royal Society. This infuriated Newton and he stopped writing to Hooke in January 1680. In 1684, Newton sent the Royal Society a manuscript, 'De Motu' ('on motion'), which took Hooke's ideas of orbital motion as its basic premise. This fact was never acknowledged by Newton. In 1687 the 'Principia Mathematica' appeared, which was a phenomenal extension of 'De Motu'. The 'Mathematical Principles of Natural Philosophy', to give it its fuller English title, would revolutionize our way of looking at and dealing with the (mechanical) world around us. The agreement between Hooke and Newton is considerable. According to Halley, it was really Hooke's correspondence with Newton which rekindled all of Newton's interest in natural philosophy. But Hooke also gave Newton the notion of the orbit (Diehl Patterson 1952; Nauenberg 1994, 2005).

If one communicates to a conical pendulum, a rotational velocity, the trajectory is now elliptic but is centred on the axis contrary to the case of the planets where the attractive point corresponds to one of the focii of the elliptical path. If the pendulum is launched without transverse initial velocity, it will not be isochroneous as believed by Galileo but its period will vary in function of the amplitude of oscillations. The surprising result obtained by Hooke (thanks to the first graphical resolution of a differential equation) is that the trajectory analogous to a planet orbit will not be a true ellipse but a rosace (Diehl Patterson 1952; Nauenberg 1994, 2005). This phenomenon, called precession, can be understood if one considers that the trajectory which is more or less circular, can be decomposed

into two separate oscillations with different amplitudes: the limiting case of the pendulum oscillating in a vertical plane corresponds to the cancelling of one of the amplitudes. Due to the nonlinearity of the pendulum, the two constitutive oscillations do not have the same periodicity which induces the precession.

In this work, we will revisit the classical works (Airy 1851*a*,*b*; Bravais 1854; Kamerlingh Onnes 1879; Routh 1898; Chazy 1930) on the precession associated to the conical pendulum with the modern viewpoint of amplitude equations formalism. In particular, one will show that some optical phenomena can be understood very easily thanks to mechanical analogues related to the conical pendulum behaviour. Concerning the vocabulary used in the paper, the term 'spherical' pendulum (rather than 'conical' pendulum) is used nowadays when both polar and azimuth angles are varying.

(*b*) *Mechanical analogue for nonlinear optical rotation*

Maker & Terhune (1964) has demonstrated several years ago the occurrence of optical effects due to an induced polarization third order in the electric field strength (Jonsson 2000). In particular, they explained quantum mechanically the precession of an elliptic light ray due to the nonlinearity of the refractive index. They have shown that the nonlinear optical precession is proportional to both the area of the elliptic path described by the polarizations vectors and the natural frequency of the phenomenon (Maker & Terhune 1964) with obvious notations

$$\Lambda_{\mathrm{NL}} = \frac{\mathrm{d}\theta_{\mathrm{NL}}}{\mathrm{d}z} = \frac{\Delta n \omega}{2c} = \frac{3}{2} \frac{\chi_{1221} n_0}{c} (|E_-|^2 - |E_+|^2)\omega. \tag{1.1}$$

In addition, either circular or linear polarization ($|E_-| = |E_+|$) avoid any precession.

We will show that this optical precession has a mechanical counterpart (Ya Zeldovich & Soileau 2004). Indeed, G. B. Airy (1851*a*,*b*) demonstrated in 1851 that a conical pendulum whose initial motion was elliptical, was compelled to process in the same direction as the oscillation of its mass (Olsson 1978, 1981; Gray *et arl.* 2004): if the length of [a conical] pendulum be l, the semi-major axis of the ellipse described by the pendulum-bob be a, and the semi-minor axis be b, then the line of the apses of the ellipse will perform a complete revolution in the time of a complete double vibration (i.e. the time of describing the ellipse) multiplied by $8l^2/3ab$.

The phenomenon was already known by experimentalists like L. Foucault who avoided the elliptical motion which would pollute experiments designed to reveal Earth's rotation which also induced an elliptical precession. Indeed, Foucault used a special set-up in order to launch his pendulum initially with a linear polarization whose plane of oscillation would only process because of the Earth's rotation and not of the Airy precession (Acloque 1981).

(*c*) *Mechanical analogue for Faraday optical rotation*

Another optical rotation can be matched with a mechanical analogue in the framework of amplitude equations formalism. In 1845, M. Faraday discovered that a linearly polarized light ray was induced to rotate by an angle θ when travelling through a medium of length L submitted to a static magnetic field B in the direction of the wavevector (Van Baak 1996; Jonsson 2000): $\theta = VBL$, where

V is the Verdet constant which depends of the medium's properties. If a mirror is placed at the end of the medium and reflects the ray, the total rotation after the round trip is twice θ and is not null as one would expect for classical optical activity in a chiral medium which demonstrates that the magnetic field is an axial vector.

Earth's rotation in Foucault experiment is also an axial vector and one would expect a similar behaviour (that is a precession induced by the rotation) of the plane of oscillation for the pendulum. We will show that this is indeed the case with the help of amplitude equations for both phenomena.

The Zeeman effect is also similar to the Foucault effect. Indeed, one of the first interpretations of this magnetic effect was given by J. Larmor with a strong analogy with mechanical effect. Indeed, in a classical model, the electron is assumed to have an elliptic trajectory around the nucleus. According to the now famous theorem of Larmor, the effect of the magnetic field can be cancelled as soon as one goes to a new frame of reference whose angular velocity is given by $\Omega = e/mB$, where e and m are the charge and the mass of the electron. Hence, the magnetic field creates an angular precession of the ellipse followed by the charged particle (Brookes Spencer 1970; Warwick 1993).

2. Amplitude equations for mechanical analogues of Faraday and nonlinear optical rotation

(*a*) *Amplitude equations for the linear Foucault pendulum*

From Newtonian or Lagrangian mechanics (Wells 1967), it is straightforward to show that the horizontal coordinates (x, y) of the pendulum satisfy the following equations:

$$\ddot{x} - 2\Omega_T \sin \lambda \dot{y} + \omega_0^2 x = 0, \tag{2.1}$$

$$\ddot{y} + 2\Omega_T \sin \lambda \dot{x} + \omega_0^2 y = 0, \tag{2.2}$$

where Ω_T is the Earth's rotation, λ the latitude and $\omega_0 = \sqrt{g/l}$ the natural pulsation of the pendulum expressed as a function of the gravity g and its length l. In addition, we used the approximations: $x, y \ll l$.

We define the complex position of the pendulum $Z = x + iy$ in order to write the equation for the motion in the horizontal plane

$$\ddot{Z} + 2i\Omega_T \sin \lambda \dot{Z} + \omega_0^2 Z = 0, \tag{2.3}$$

whose solution is

$$Z = [A_0 \exp(i\omega_0 t) + B_0 \exp(-i\omega_0 t)]\exp(-i\Omega_T \sin \lambda t). \tag{2.4}$$

The corresponding amplitude equations for the two circular polarization with Earth's rotation are $\dot{A} = i(\omega_0 - \Omega_T \sin \lambda)A$ and $\dot{B} = -i(\omega_0 + \Omega_T \sin \lambda)B$. The Foucault precession can be interpreted as a kind of Doppler effect for the two circular polarizations as the frequencies of oscillation are modified by the Earth's rotation in the same way (i.e. with a negative sign): the total frequency of the B-polarization is increased compared to the one of the *A*-polarization which is diminished. The same interpretation is valid for the Zeeman effect which implies

a doubling of the spectrum lines of an incandescent vapour (Brookes Spencer 1970; Warwick 1993).

(b) *Amplitude equations for the nonlinear conical pendulum*

We now solve the equation for the conical pendulum without rotation taking into account its intrinsic nonlinearity (the Hooke pendulum). The dot corresponds to the partial derivative with respect to the dimensionless time $t^* = t\omega_0$.

From Lagrangian mechanics, one deduces the nonlinear equation for the complex position with the dimensionless variables as follows.

The Lagrangian for a conical pendulum is

$$L^* = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{h}^2) - mgh. \tag{2.5}$$

We introduce dimensionless variables

$$X = \frac{x}{l}, \quad Y = \frac{y}{l}, \quad H = \frac{h}{l}, \quad t^* = t\sqrt{\frac{g}{l}}. \tag{2.6}$$

The spherical constraint leads to

$$H = 1 - \sqrt{1 - (X^2 + Y^2)} \simeq \frac{X^2 + Y^2}{2} + \frac{(X^2 + Y^2)^2}{8}, \tag{2.7}$$

with $X \ll 1$ and $Y \ll 1$ and where we keep the fourth-order term in the development of the height since it is necessary to balance the fourth-order term in the development of the vertical velocity. The other terms are of the second order.

Now, one can use the dimensionless complex position $Z = X + iY$ (with $|Z|^2 = X^2 + Y^2$) and Lagrangian

$$L = \frac{L^*}{mgl} = \frac{1}{2}\left[|\dot{Z}|^2 + \frac{1}{4}(\dot{Z}\overline{Z} + Z\dot{\overline{Z}})^2\right] - \frac{1}{2}|Z|^2 - \frac{1}{8}|Z|^4, \tag{2.8}$$

with the Euler–Lagrange equation in order to get (after some algebra)

$$\frac{1}{2}(1 + \frac{|Z|^2}{2})\ddot{Z} + \frac{1}{4}Z^2\ddot{\overline{Z}} + \frac{1}{2}Z + \frac{1}{4}|Z|^2 Z + \frac{1}{2}|\dot{Z}|^2 Z = 0. \tag{2.9}$$

We can rewrite the last equation with its complex conjugate in terms of a system with the help of matrices which we invert in order to have the second-order equation in time which describes the evolution of the complex position

$$\ddot{Z} + Z - \tfrac{1}{2}|Z|^2 Z + |\dot{Z}|^2 Z = 0, \tag{2.10}$$

and which we solve with

$$Z = A\exp(it) + B\exp(-it). \tag{2.11}$$

As usual within the amplitude equation formalism, the second derivatives in the amplitude A and B are neglected with respect to the first derivatives as we look for slow varying evolutions. In addition, we have used the fact that the modulus of the complex position is small, as we are close to verticality, in order to obtain the evolution equation.

A more direct way of obtaining the evolution equation for Z is to decompose the motion in X and Y coordinates and then infer the equation for Z. Indeed, the motion of the conical pendulum is just the sum of the motion of two pendulums oscillating in two perpendicular planes. If one denotes θ the angle with respect to the vertical of the pendulum oscillating in the X direction, one obviously has

$$\ddot{\theta} + \sin\theta = 0. \tag{2.12}$$

With $X = \sin\theta$, we get

$$\ddot{X} = -X\sqrt{1-X^2} - \frac{\dot{X}^2}{1-X^2}X. \tag{2.13}$$

We end up with

$$\ddot{X} + X - \tfrac{1}{2}X^3 + \dot{X}^2 X = 0, \tag{2.14}$$

where we used $X \ll 1$. Now to deduce the equation for Z is very simple. One replaces X by Z in the former equation by remembering that Z is a complex variable and that the Z equation must respect the symmetry $Z \to Z e^{i\phi}$ corresponding to time translation in the complex plane. In particular, Z^3 breaks the symmetry but not $|Z|^2 Z$.

We deduce a set of first order in time coupled amplitude equations for both A and B:

$$\dot{A} = iA(\alpha|A|^2 + \beta|B|^2) \tag{2.15}$$

and

$$\dot{B} = -iB(\alpha|B|^2 + \beta|A|^2), \tag{2.16}$$

with $\alpha = 1/4$ and $\beta = -1/2$. One obtains a rosace-like trajectory which can be interpreted as the sum of two oscillations with different amplitude.

Now, one infers that the moduli of the amplitudes are constant ($|A| = A_0 = R$, $|B| = B_0 = S$) because

$$\partial_{t^*}|A|^2 = 0 \quad \text{and} \quad \partial_{t^*}|B|^2 = 0. \tag{2.17}$$

The amplitude equations are now simpler:

$$\dot{A} = iA(\alpha|A_0|^2 + \beta|B_0|^2) \tag{2.18}$$

and

$$\dot{B} = -iB(\alpha|B_0|^2 + \beta|A_0|^2). \tag{2.19}$$

One notices that these equations are invariant by rotation:

$$A \to A \exp i\varphi \quad \text{and} \quad B \to B \exp -i\varphi, \tag{2.20}$$

and they can be solved with: $A = R\exp(i\Phi) = R\exp(i\omega_A t^*)$, and $B = S\exp(-i\Psi) = S\exp(-i\omega_B t^*)$ which leads to: $\omega_A = \alpha R^2 + \beta S^2$ and $\omega_B = \alpha S^2 + \beta R^2$.

Then, one infers the angle of precession of the elliptic path:

$$\Lambda = \frac{\Phi - \Psi}{2} = \frac{\omega_A - \omega_B}{2}t^*, \tag{2.21}$$

that is

$$\Lambda = \frac{\alpha - \beta}{2}a^* b^* t^*, \tag{2.22}$$

with $a^* = R + S (b^* = R - S)$ the major (minor) semi-axis of the elliptic path. One concludes that the angular velocity of precession is proportional to the surface of the ellipse ($\pi a b$):

$$\Omega_{\mathrm{p}}^* = \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \frac{\alpha - \beta}{2} a^* b^* = \frac{3}{8} a^* b^* = \frac{3}{8}(R^2 - S^2), \tag{2.23}$$

as was discovered by Airy with the dimensional form

$$\Omega_{\mathrm{p}} = \frac{3}{8}(R^2 - S^2)\omega_0 = \frac{3}{8}\frac{ab}{l^2}\sqrt{\frac{g}{l}}. \tag{2.24}$$

Our derivation is mathematically less complicated than the original one of Airy. Indeed, starting from angular momentum conservation, energy conservation and using the spherical constraint, he derived a fourth-order differential equation which he solved at each order (Airy 1851*a,b*). Moreover, from the physical point of view, the amplitude equation formalism points out both polarizations which are modified by the nonlinearity and the rotation as we will show in the rest of the paper.

(*c*) *Amplitude equations for the Hooke's pendulum with Foucault's rotation*

Now, we would like to solve the equation for the conical pendulum with both intrinsic nonlinearity and external rotation:

$$\ddot{Z} + Z - \tfrac{1}{2}|Z|^2 Z + |\dot{Z}|^2 Z = 0, \tag{2.25}$$

with the following ansatz: $Z = W \exp(\mathrm{i}\Omega t)$ which consists of changing from the rotative frame of reference to the stationary one. The nonlinear Foucault's equation for W is

$$\ddot{W} + 2\mathrm{i}\Omega\dot{W} - \Omega^2 W + W - \tfrac{1}{2}|W|^2 W + |\dot{W}|^2 W - \mathrm{i}\Omega\dot{W}|W|^2 + \mathrm{i}\Omega W^2\overline{\dot{W}}$$

$$+ \Omega^2|W|^2 W = 0, \quad (2.26)$$

where we recognize a Coriolis-like term ($2\mathrm{i}\Omega\dot{W}$) and a centrifugal-like term ($-\Omega^2 W$).

Now, one solves the nonlinear equation for W by introducing the following solution mimicking the linear resolution:

$$W = A \exp(\mathrm{i}t) + B \exp(-\mathrm{i}t). \tag{2.27}$$

One ends up with the equation for the A-polarization:

$$\dot{A} = -\frac{\mathrm{i}A}{2}[\delta_0 + \delta_1|A|^2 + \delta_2|B|^2], \tag{2.28}$$

with $\delta_0 = 2\Omega + \Omega^2$, $\delta_1 = -(1/2 + 2\Omega + \Omega^2)$ and $\delta_2 = 1 + 2\Omega - 2\Omega^2$.

If $\Omega = 0$, one has: $\delta_0 = 0$, $\delta_1 = -1/2$ and $\delta_2 = 1$. One recovers the amplitude equation for the A-polarization of the simple nonlinear conical pendulum.

If $\Omega \ll 1$ (i.e. $\Omega \ll \Omega^2$), one obtains

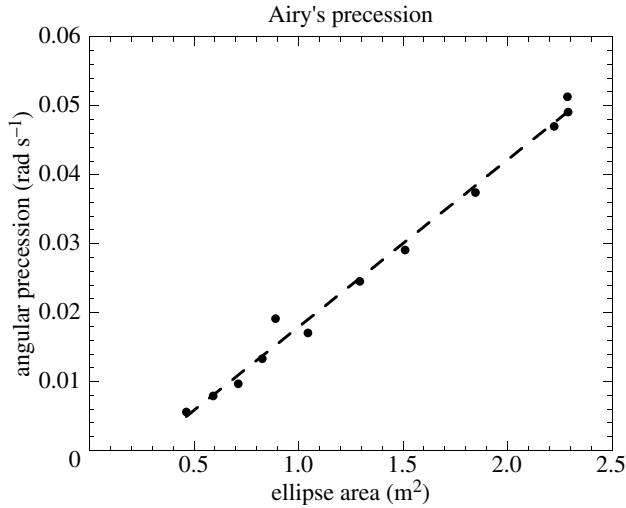$$\dot{A} = -\frac{\mathrm{i}A}{2}[+2\Omega - \frac{1}{2}|A|^2 + |B|^2]. \tag{2.29}$$

Figure 1. Experimental results: angle of precession of the elliptic path in function of the ellipse area.

The equation for the B-polarization becomes

$$\dot{B} = -\frac{iB}{2}[\gamma_0 + \gamma_1|A|^2 + \gamma_2|B|^2], \qquad (2.30)$$

with $\gamma_0 = 2\Omega - \Omega^2$, $\gamma_1 = 1/2 - 2\Omega + \Omega^2$ and $\gamma_2 = -1 + 2\Omega + 2\Omega^2$.

If $\Omega = 0$, one has: $\gamma_0 = 0$, $\gamma_1 = 1/2$ and $\gamma_2 = -1$. One recovers the amplitude equation for the B-polarization of the simple nonlinear conical pendulum.

If $\Omega \ll 1$ (i.e. $\Omega \ll \Omega^2$), one obtains

$$\dot{B} = -\frac{iB}{2}[+2\Omega + \frac{1}{2}|B|^2 - |A|^2]. \qquad (2.31)$$

One notices that for the two circular polarizations, the axial feature of the rotation vector leads to the same sign for the terms proportional to Ω whereas the nonlinear terms are of opposite sign in the amplitude equations for A and B. Hence, this illustrates once again the fact that the Coriolis effect in mechanics is similar to the Faraday rotation in optics.

3. Experiments

In order to verify Airy's precession law, we built a conical pendulum with a hollow globe in brass of diameter 18 cm which we filled with lead beads: the mass is of the order of 20 kg. We suspended it from a steel cable of length 2.72 m. Hence, the small angle approximation is fulfilled with such a length compared to the horizontal projection of the trajectory. In addition, air motion will not perturb such a weight too much.

For the measurements, one draws on the ground two axes with a predefinite angle (say 40°). Then, one launches the pendulum. When its path of oscillation crosses the first axis parallel to it, we start the chronometer and we stop it when

the pendulum reaches the second axis. At the same time, one marks with chalk the lengths of the major axis and the minor axis with a vertical plane, which allows us to draw on the ground the projection of the tangents to the trajectory perpendicularly to both axis of the ellipse. Hence, one has the area and the time during the precession between the two axes of references.

The comparison between theory and experiments is in good agreement with respect to the ratio between the angular precession and the elliptic area (figure 1): 2.5×10^{-2} for the experiments and for the theory 2.6×10^{-2}. This coincidence is quite successful despite the numerous effects which would lead to disagreement (Olsson & Nelson 1986): elasticity/torsion of the cable, air drag, role of the fixation, second-order effects.

4. Conclusions

The amplitude equation formalism is a powerful technique in order to display the universal behaviour of apparently disconnected physical phenomena. By revisiting the seminal works of Robert Hooke on the conical pendulum, we were able to show its close relationships with the behaviour of the polarization of light waves inside nonlinear media in presence of a magnetic field. We hope that our study will clarify the physical pictures associated with such optical processes.

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