Wave-Current Interaction as a Spatial Dynamical System: Analogies with Rainbow and Black Hole Physics

Jean-Charles Nardin, Germain Rousseaux, and Pierre Coullet
Laboratoire J.-A. Dieudonné, Université de Nice-Sophia Antipolis, UMR CNRS-UNS 6621, Parc Valrose, 06108 Nice Cedex 02, France, European Union
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We study the hydrodynamic phenomenon of waves blocking by a countercurrent with the tools of dynamical systems theory. We show that, for a uniform background velocity and within the small wavelength approximation, the stopping of gravity waves is described by a stationary saddle-node bifurcation due to the spatial resonance of an incident wave with the converted ''blueshifted'' wave. We explain why the classical regularization effect of interferences avoids the height singularity in complete analogy with the intensity of light close to the principal arc of a rainbow. The application to the behavior of light near a gravitational horizon is discussed.

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Water waves propagating on a countercurrent are characterized by a significant increase of their height, and the resulting rogue waves are a danger for ships sailing across the interaction zone. The amplification mechanisms are refraction and reflection. On the other hand, wave breakers made of bubble curtains producing surface currents are used to stop gravity waves in marine applications [1]. Moreover, the influence of currents on sediment transport modifies the mass budget due to the water waves and is the subject of investigations for coastal engineering [2]. Here we derive the normal form associated with the bifurcation due to the spatial resonance of incoming waves and converted ones leading to waves blocking. We display the control parameter and discuss the analogies with rainbow and black hole physics. In particular, we derive the universal scaling exponent of the diverging energy of water waves close to the blocking boundary due to the countercurrent in perfect analogy to the caustic for light intensity of the rainbow. The possibility to measure the classical analogue of the Hawking temperature associated to the quantum radiation of black holes is underlined through the design of wave-current interaction experiments in the laboratory [3].

Gravity waves in the presence of a uniform current are described by the following dispersion relation: $(\omega - \mathbf{U}\cdot\mathbf{k})^2 = gk\tanh(kh)$, where $\omega/2\pi > 0$ is the frequency of the wave and k the algebraic wave number [4–12]. g denotes the gravitational acceleration of the Earth at the water surface, $U < 0$ is the constant velocity of the background flow, and h is the height of the water depth. The flow induces a Doppler shift of the pulsation ω. The dispersion relation is usually solved by graphical means (Fig. 1). One important point is that there exist four branches $\pm\sqrt{gk\tanh(kh)}$. However, the fluid community focused so far only on the positive wave number k (except Peregrine in the case $U > 0$ [4]), whereas the relativistic community has drawn recently the attention to the negative k since the equation which describes the propagation of water waves riding a current in the long wavelength approximation is strictly the same as for the propagation of light near a Schwarzschild black hole [13]. In particular, negative wave numbers k with both positive and negative relative frequencies $\omega' = \omega - Uk$ in the frame of the current can appear by mode conversion. Depending on the parameters, two or three branches are intercepted by the straight line $\omega - Uk$ with a positive slope since only countercurrents with $U < 0$ are considered in this work.

A maximum of four solutions is possible (Fig. 1): two with positive k and two with negative k. Concerning the positive solutions, one (k_I) corresponds to the incident wave, and the other one (k_B) describes a wave which is ''blueshifted'' as its wave number is larger than the incident wave. The blue wave is often wrongly confounded with a ''reflected'' wave. Indeed, the former have a positive wave number but a negative group velocity: The slope of the straight line is superior to the tangent to the positive square-root branch at k_B. In addition, the phase velocity of the blueshifted wave is positive such that its crests move in the opposite direction to the countercurrent seen from the

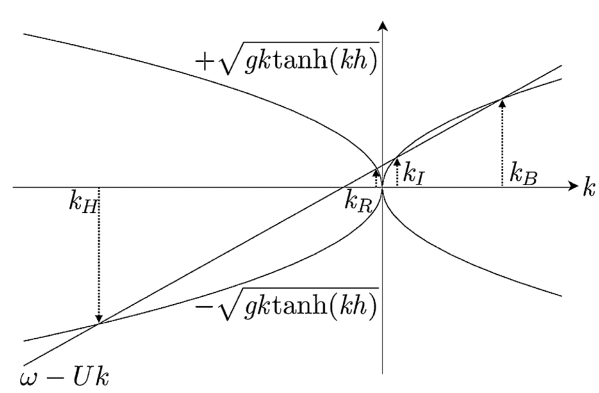

FIG. 1. Graphical solutions of the dispersion relation.

laboratory rest frame. When the countervelocity increases, k_I and k_B get closer until both wave numbers merge: This defines wave blocking where the total group velocity vanishes. Concerning the negative solutions, one (k_R) corresponds to the retrograde wave (already present without a current) with a positive frequency in the current frame, whereas the other (k_H), which has a negative relative frequency, is called a ‘‘negative energy wave’’ [9]. The latter is interpreted by the relativistic community as the classical ingredient leading to the spontaneous radiation of a black hole, the Hawking effect, since a creation operator in quantum physics corresponds to a negative relative frequency in classical physics [3,13]. Recently, we observed experimental indications for such a mode conversion from positive relative frequencies to negative ones [3].

As described in the literature [4,6–8,10–12], for wavelengths small compared to the water height, we can solve easily the resulting approximate dispersion relation, which is a quadratic polynomial in k: $(\omega - Uk)^2 \simeq gk$. It was found that the blocking velocity is $U^* = -g/(4\omega)$ since the two solutions of the equation are

$$k = \frac{2U\omega + g \pm g\sqrt{1 + \frac{4U\omega}{g}}}{2U^2}. \tag{1}$$

The disappearance of k_I and k_B at the blocking point happens where the straight line $\omega - Uk$ is tangent to the positive square-root branch $\sqrt{gk\tanh(kh)}$. In the wave-blocking problem, the dispersion relation provides a quadratic polynomial in k for the small wavelength approximation [6]. Indeed, one keeps graphically a curved branch which is intercepted twice by the relative pulsation ω'. Hence, if we can zoom on the blocking point, we should be able to write the canonical form of the underlying dynamical system. The problem is stationary and so is the canonical form. The dispersion relation is written $\omega^2 - 2\omega Uk + U^2k^2 \simeq gk$. Now the process to obtain the normal form of the tangent bifurcation features two steps: translation and renormalization of the parabola embedded in the quadratic polynomial. We translate the wave number in order to suppress the linear term in the polynomial by introducing a new wave number $k' = k - A$. The dispersion equation becomes $\omega^2 + k'(2U^2A - 2U\omega - g) - A(2\omega U + g) + U^2A^2 + U^2k'^2 = 0$. A local description of the dynamics close to the blocking point implies $A = (2U\omega + g)/2U^2$. However, the normal form should be written with dimensionless parameters. Let us compute A close to the bifurcation, and we find

$$A \overset{U\to U^*}{\longrightarrow} k^* \simeq \frac{4\omega^2}{g}. \tag{2}$$

As a consequence, the normal form describing the spatial resonance associated to the merging of the incident wave with the blueshifted one (leading to interferences because of their opposite group velocities) is written [after renormalization by introducing the dimensionless wave number $\kappa = (k - k^*)/k^*$]

$$\mu - \kappa^2 = 0, \tag{3}$$

where $\mu = -4(U - U^*)/U^*$ is the so-called control parameter of a saddle-node bifurcation (the system loses simultaneously two real solutions at the bifurcation) which does describe wave blocking by a countercurrent. Wave blocking appears by changing either the countercurrent velocity (which changes the slope of the straight line in the graphical resolution) or the pulsation (which shifts vertically the straight line).

Similarly to optical phenomena such as the rainbow, the wave-blocking boundary is a caustic where the water waves pile up. The height increases and would diverge as shown numerically for deep water in Ref. [14]. However, usually, a classical regularization process avoids the height singularity by either wave breaking before reaching the stopping point or by mode conversion in the capillary range at the blocking point as reported by Badulin, Pokazeev, and Rozenberg [7]. In optics, the catastrophe is smoothed by the appearance of a classical process such as interferences or diffraction [15]. Let us recall, for example, that, in order to soften the singularity of the light intensity distribution, interference fringes appear in the vicinity of the main arc in supernumerary rainbows [16]. These fringes are described by the Airy function [17] and are due to the self-wrapping of the wave front on itself because of the reflection within water drops [16]. Two ingredients are necessary to get an Airy equation: reflection on the side of the drop and diffraction due to the wave-front narrowing within the drop. In order to quantify the analogy between the rainbow and wave-current interaction, we will compute for the first time the scaling law for the divergence near the wave-blocking boundary. Let us recall that the appearance of the rainbow caustic is due to the existence of a minimum of deviation of the sun rays impacting on a spherical drop. One introduces usually the Taylor expansion of the so-called deflection function $D(i)$ close to its minimum as a function of the angle of incidence i and its associated minimum [16]:

$$D(i) \simeq D_{\min} + \frac{1}{2}\frac{\partial^2 D}{\partial i^2}(i - i_{\min})^2. \tag{4}$$

Then one can show that the light intensity $I(D)$ will diverge as it is focused around $D_{\min}$ according to $I(D) \approx 1/(\partial D/\partial i) \approx \Theta(D - D_{\min})(D - D_{\min})^{-1/2}$, where Θ is the Heaviside step function [15,16]. This result is derived from geometrical optics only. The square-root behavior of the singularity is typical of a fold caustic [15] (a stationary saddle-node bifurcation in our case) which we will prove as well for the wave-blocking boundary in the depth water approximation. The fact that the intensity grows as the inverse square root of the ‘‘distance’’ to the caustic is in contrast with the simple inverse variation for the caustic associated to a parabolic or elliptic mirror which does not display a bifurcation as the incident angle varies.

From the Doppler formula, it is well known that the phase velocity c_φ for water waves propagating on a countercurrent in deep water is given by [8,14]

$$c_\varphi = -4U^*\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{U}{U^*}}\right). \quad (5)$$

The conservation of the wave action flux Ec_g/ω for water waves of group velocity $c_g = \partial\omega/\partial k$ and energy $E = 1/2\rho g a^2$ (which is the analogue of the light intensity I) leads to [8,14]

$$\frac{E}{E_{\text{far}}} = \left(\frac{a}{a_{\text{far}}}\right)^2 = \frac{16U^{*2}}{(2U + c_\varphi)c_\varphi}, \quad (6)$$

where a is the wave amplitude. One refers to E_{far} (a_{far}) as the incoming energy (amplitude) of the water waves far from the current. By recalling that the velocity U is close to its blocking value U^* in the denominator of Eq. (6) and using Eq. (5), we get $E/E_{\text{far}} \simeq 4\sqrt{U^*/(U^* - U)}$. Hence, the energy of water waves diverges close to the wave-blocking boundary according to the following scaling law (expressed with our control parameter μ): $E(\mu) \approx 1/(\partial\omega/\partial k) \approx \Theta(\mu)\mu^{-1/2}$ in complete analogy to the caustic of the rainbow for light intensity. The scaling exponent is universal and is the consequence of the saddle-node behavior of a fold catastrophe. In actual experiments, the velocity U varies with the position x [3]. This would lead us with the following scaling law for the amplitude as a function of the distance to the hydrodynamic caustic x^* assuming a constant velocity gradient: $a(x) \approx \Theta(x - x^*)(x - x^*)^{-1/4}$, which is similar to the Green's law for wave-bottom interaction with a linear slope [8]. The authors of Refs. [10,11] claimed that their recent measurements of the water height envelope near the blocking boundary is described by an Airy function We claim that it would be similar to the light intensity in the rainbow due to the interferences regularization process [16]. The Airy equation is usually derived by symmetry arguments and/or a Lagrangian formulation (see [11] for a recent survey). It was introduced initially by Smith as a basic mechanism leading to the generation of extreme waves [5]. Smith derived a modified nonlinear Schrödinger equation by essentially taking the Fourier transform of the dispersion relation. It is worth recalling here that Basovich and Talanov [6] derived an Airy equation through the Fourier transform of the Taylor expansion up to the second order of the velocity close to the blocking velocity which is reminiscent of our dynamical system approach. However, neither Smith nor Basovich and Talanov acknowledged the existence of the saddle-node bifurcation, and they did not derive the dimensionless control parameter.

We are looking now for an equation describing the time and space evolution of the wave envelope $A(x, t)$ propagating in the "linear potential well" $\mathcal{V} = \mathcal{V}(x) = \nu(x - x^*)$ due to the longitudinal space variation of the velocity field encountered by the waves. The equation must be characterized by the following symmetries: space translation ($x \to x + x_0$) and separately time translation ($t \to t + t_0$) due to the homogeneity of space and time ($A \to Ae^{i\phi}$). One expands the time derivative of the amplitude as a Landau-type equation $\partial A/\partial t = f(A, \bar{A})$, where the overbar means complex conjugate. From the dispersion relation, one must keep the dispersive effect due to the quadratic term in k. The oscillation is encoded in the first-order time derivative. The simplest linear amplitude equation which is reversible in both time and space in order to allow propagation ($t \to -t$ and simultaneously $x \to -x$ lead to $A \to \bar{A}$) is the linear Schrödinger equation $\partial A/\partial t = i\mathcal{V}A - i\beta\partial^2 A/\partial x^2$. If the potential $\mathcal{V}$ was constant, we could absorb it in a new definition of the amplitude $B = Ae^{i\mathcal{V}t}$. As we will see, the potential breaks the space translation symmetry. A constant velocity field would break the space reflection; hence, a first-order spatial derivative should be added. The partial time derivative (∂_t) should be changed to a total derivative ($\partial_t + c_g\partial_x$), with $c_g = \partial\omega/\partial k$ the group celerity that is by definition the velocity of the wave envelope. We took c_g as the real coefficient in front of the spatial derivative since it has the dimension of a velocity, and we look for the time evolution of the envelope that is of the energy carrier. Without current and due to the Galilean symmetry, one could absorb it in a new definition of time switching to the frame of reference of the wave envelope. However, the countercurrent selects a particular frame, and the first-order spatial derivative must be taken into account. We end up with a Schrödinger-like equation:

$$\frac{\partial A}{\partial t} + c_g\frac{\partial A}{\partial x} = i\mathcal{V}A - i\alpha\frac{\partial^2 A}{\partial x^2} - i\beta|A|^2 A - \gamma A, \quad (7)$$

where α, β, and γ are all real because of the symmetries. For completeness, we add the nonlinear term and a dissipative term corresponding to viscous effects both consistent with the rotational invariance. Assuming time independence, neglecting both dissipation and nonlinearity, we get

$$\nu(x - x^*)A - \alpha\frac{d^2 A}{dx^2} = 0, \quad (8)$$

the Airy equation where $\nu = k^* dU/dx$ measures the spatial variation of the velocity close to the blocking line. The first-order spatial derivative vanishes at the bifurcation since $c_g = 0$ when the blocking occurs. Around the bifurcation, the group velocity no more vanishes, and the spatial derivation would be equivalent to the multiplication by k^* which is the same as assuming a null group velocity and a linear potential well as in wave-bottom interaction. The wavelength λ should scale with the typical length on which the flow gradient varies in order to have blocking; shorter waves will pass like the capillary waves after a mode conversion. The dimensionless parameter $(\lambda dU/dx)/c_g$ defines the condition of application of the amplitude equation formalism. Therefore, contrary to previous derivations found in the literature [11], the envelope of the wave ''reflected'' by a countercurrent (drop's interface in rainbow physics) and ''diffracted'' by its gradient (focusing in the confined spherical geometry of the drop) is described by an Airy function provided that the velocity gradient is constant and independent of the longitudinal position. Hence, the measurements of Chawla and Kirby [10] and Suastika [11] should be analyzed in the light of this strong constraint. Indeed, if the velocity profile is not linear with the distance to the blocking point, the wave envelope will not be described by an Airy function, and we can infer that some of the discrepancies reported in their measurements with respect to the theory may be related to the relaxation of this constraint in addition to the obvious influence of nonlinearities due to the large amplitude of incoming waves. As a naive mechanical picture, the wave-current interaction can be mapped onto the propagation of a train of small disturbances on a linear chain of coupled pendula (see [18]) with varying lengths and reflected on a wall. It is straightforward to linearize the corresponding sine-Gordon equation to get a Schrödinger-like equation and to add a linear variation of the length of each oscillator along the chain in order to mimic a weak linear flow gradient or bottom slope.

If quantum phenomena are allowed by using flowing superfluids or Bose-Einstein condensates, quantum radiation similar to the Hawking radiation of black holes or to the Schwinger effect can regularize as well the classical catastrophe [19]. Indeed, the countercurrent is analogous to a white hole in astrophysics (the time reversal of a black hole, that is, a gravitational fountain) and the water waves translate into light waves [20,21]. Inversely to the redshift of black hole, white holes are characterized by blueshifting as the wavelength decreases close to the event horizon, which is analogous to the blocking boundary for water waves since it is defined by the equality between the light velocity c_L and the gravitational escape velocity $v_{\rm lib} = \sqrt{2GM/R}$, where G is the Newton constant, M the mass of the white hole, and R its radius [3]. The velocity gradient plays the role of the surface gravitation at the horizon [20,21]. The Hawking temperature is directly proportional to the latter, that is, to our control parameter μ if experiments were performed with superfluids or Bose-Einstein condensates. We can guess that the measurement of the water height is an indirect way to infer the velocity gradient that is the Hawking temperature working with classical fluids.

An interesting perspective of our work would be to derive the saddle-node bifurcation and the associated control parameter when taking into account the effect of surface tension which introduces a second blocking velocity as reported by Badulin, Pokazeev, and Rozenberg [7]. The mode conversion into these smaller scales for the fluid system is reminiscent of the so-called trans-Planckian problem in black hole physics [20]. Indeed, due to dispersive effects close to the event horizon, the Hawking radiation could be avoided because of quantum gravity regularization processes occurring below the Planck scale which would be a cutoff for the light wavelength similarly to the capillary length or the mean free path.

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