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On the "bead, hoop and spring" (BHS) dynamical system

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Abstract Here, we make the theoretical and numerical analysis of the non-linear equation describing the evolution of the "bead, hoop and spring" (BHS) dynamical system derived by Ochoa and Clavijo in (Eur. J. Phys. 27:1277–1288, 2006). In particular, we solve by standard techniques of non-linear physics an approximation of their equation neglecting the centrifugal effect before giving a more mathematical and exact treatment. The analogy with phase transitions is underlined. We point out the existence of finite-time singularities in the phase-space and we derive a criterion for possible oscillations.

Keywords Non-linear oscillations · Duffing-like system · Singularity

1 Introduction

Phase transitions in the mean-field approximation bear strong resemblance with mechanical bifurcations. In particular, one speaks of critical behavior, order parameter or exponents in both problems [1–5].

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Ochoa and Clavijo [1] have derived recently the equation of motion for the bead, hoop and spring problem (the dot stands for a partial derivative with respect to time t):

$$\ddot{z} + \left(\frac{z}{R^2 - z^2}\right)\dot{z}^2 + \left(\frac{2k}{mR^2} - \frac{\omega^2}{R^2}\right)z^3 + \frac{2kr_0}{mR^2} z\sqrt{R^2 - z^2} - \left(\frac{2k}{m} - \omega^2\right)z = 0 \quad (1)$$

and they solved it numerically.

The mechanical system is made of two beads of mass m linked by a spring which is bound to slide on a horizontal loop of radius R (see Fig. 1). The latter is put into rotation with angular velocity ω around a horizontal axis denoted by z. More precisely, the two beads, sliding freely on the ring, are connected by a light straight spring (that remains straight not just under tension but also under compression) and which is constrained to be perpendicular to the diameter of the ring that coincides with the fixed horizontal rotation axis. The beads have the same z-coordinate. This is a crucial constraint that brings down the degrees of freedom to one and also makes gravity irrelevant since it constrains the center of mass of the beads to the horizontal rotation axis.

The centrifugal force tends to extend the distance between the beads whereas the elasticity of the spring, k, counteracts this effect. Each variation of the distance $2r$ between both beads whose initial value is $2r_0$ translates into a changing position of the center

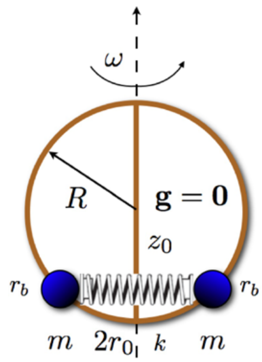

Fig. 1 Scheme of the experimental setup.

of mass of the two beads along the horizontal axis. There are two geometrical constraints, $r_0 < R$ and $r = \pm\sqrt{R^2 - z^2}$. $z = 0$ corresponds to the position of the mass center of the beads when they are at a distance $2R$.

Ochoa and Clavijo have introduced a peculiar angular velocity $\omega_c^2 = \frac{2k}{m}(1 - \frac{r_0}{R})$ which allows to interpret the bead, hoop and spring problem as a critical phenomenon in analogy with statistical mechanics [1]. In our previous comment [6], we derived the amplitude equation (corresponding to the cubic Landau equation for the analogue magnetic system) of the resulting non-linear oscillator, provided that $\omega > \omega_c$. As a matter of fact, under this last constraint, the mechanical system is perfectly similar to the Ising model for ferromagnetism and enters into the universal class of second-order phase transitions in the mean-field approximation [1]. The order parameter is the position (magnetization) in the mechanical (magnetic) system. Precisely, we explained the numerical phase-space (Fig. 8 in [1]) as essentially the one corresponding to a cubic non-linear oscillator.

We would like to treat, for the first time, the other case when $\omega < \omega_c$. In particular, a complete analytical solution of the non-linear problem of the BHS system has not been considered before in the literature.

The paper is organized as follows. Section 2 deals with the conventional resolution of an approximation of the BHS problem, namely, a Duffing-like equation which allows us to explain the qualitative behaviors reported in the numerical simulations of Ochoa and Clavijo [1]. Section 3 is concerned with some new numerical illustrations of the phase-space, corresponding to the complete BHS problem. Finally, we give a novel and thorough analytical treatment in Sect. 4 where regimes with or without oscillations are displayed.

2 Simplified case without the centrifugal term in the small-amplitude approximation

We use dimensionless variables $z = Z \times R$ and suppose for the moment that $\Omega_0^2 = -\omega_0^2 = \frac{2k}{m} - \omega^2 > 0$. The evolution equation becomes:

$$\ddot{Z} + \left(\frac{Z}{1 - Z^2}\right)\dot{Z}^2 + \Omega_0^2 Z^3 + \frac{2kr_0}{mR} Z\sqrt{1 - Z^2} - \Omega_0^2 Z = 0. \tag{2}$$

Now, we assume that the displacements are small, $Z \ll 1$, and we introduce $\Omega_1^2 = \omega_1^2 = \frac{2kr_0}{mR}$ to get:

$$\ddot{Z} - (\Omega_0^2 - \Omega_1^2) Z + \left(\Omega_0^2 - \frac{\Omega_1^2}{2}\right) Z^3 + Z\dot{Z}^2 \simeq 0. \tag{3}$$

At this stage, we can relax the stronger constraint $\omega^2 < \frac{2k}{m}$ for a weaker one $\omega^2 < \frac{2k}{m}(1 - \frac{r_0}{R}) = \omega_c^2$, that is, $\Omega_1^2 < \Omega_0^2$: we do recover the critical angular velocity of Ochoa and Clavijo.

The former equation of motion is difficult to solve because of the centrifugal term $Z\dot{Z}^2$. We will make the strong assumption that it is negligible. This will allow us to find approximate solutions which illustrate the numerical solutions obtained by Ochoa and Clavijo, as displayed in their Fig. 7 [1]. Later, we will provide a numerical phase-space analysis taking into account the influence of the centrifugal term before giving a complete mathematical resolution.

Without the centrifugal term, the equation of motion writes (we recall that $\Omega_0^2 > \Omega_1^2 > \Omega_1^2/2$):

$$\ddot{Z} \simeq (\Omega_0^2 - \Omega_1^2) Z - \left(\Omega_0^2 - \frac{\Omega_1^2}{2}\right) Z^3. \tag{4}$$

With $\tau = \sqrt{(\Omega_0^2 - \Omega_1^2/2)}t$ and the new temporal derivative $\partial()/\partial\tau = ()_\tau$, we find the so-called "reversible pitchfork" equation or Duffing equation:

$$\frac{\partial^2 Z}{\partial \tau^2} \simeq \mu Z - Z^3 \tag{5}$$

with $\mu = (\Omega_0^2 - \Omega_1^2)/(\Omega_0^2 - \Omega_1^2/2) > 0$. The symmetries of this equation are: $\tau \to -\tau$ and $Z \to -Z$. The three stationary solutions are: $Z = 0$ and $Z = \pm\sqrt{\mu}$. We do recover the exponent $1/2$ as is usual in the mean-field theory of phase transition. The neglected

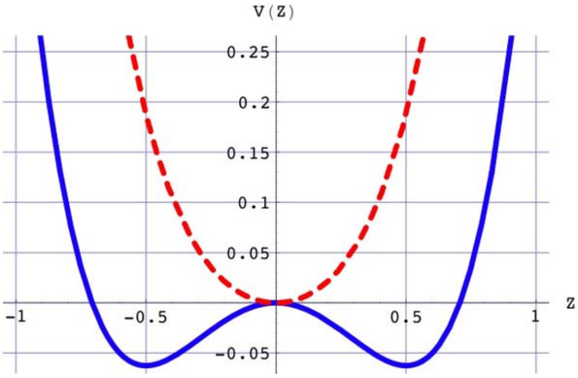

Fig. 2 Potential $V(Z) = Z^4/4 - \mu Z^2/2$ of the approximate equation for small amplitudes and without the centrifugal term: full line with $\mu = +1$ and dashed line with $\mu = -1$

centrifugal term ZZ_τ^2 would not change these symmetries and solutions.

The behavior of the solution of the "reversible pitchfork" equation can be easily understood if one rewrites it in the form of a Newton equation for a particle in a potential well:

$$\frac{\partial^2 Z}{\partial \tau^2} = -\frac{\partial V}{\partial Z} \tag{6}$$

with $V(Z) = Z^4/4 - \mu Z^2/2$ (see Fig. 2). As μ is positive, the potential has the shape of a concave Mexican hat with a maximum at $Z = 0$, two mimima at $Z = \pm\sqrt{\mu}$, and two divergences toward infinity for large $|Z|$. The function $V(Z)$ is even.

Now, the Newton equation can be cast into the form of a dynamical system:

$$\begin{aligned} \frac{\partial Z}{\partial \tau} &= v, \\ \frac{\partial v}{\partial \tau} &= \mu Z - Z^3. \end{aligned} \tag{7}$$

The fixed points are such that $\partial Z/\partial \tau = 0$ and $\partial v/\partial \tau = 0$, which implies $Z = 0$ or $Z = \pm\sqrt{\mu}$. In the phase-space $(Z, \partial Z/\partial \tau)$, the fixed points have the following coordinates: $F_0 = (0, 0)$, $F_1 = (\sqrt{\mu}, 0)$ and $F_2 = (-\sqrt{\mu}, 0)$.

First, we use a linear theory around the fixed point F_0 in order to obtain an approximate behavior of the vertical displacement Z. We focus on the solution B of Fig. 7(e) of [1] in the close vicinity of F_0 (later in Sect. 2 a complete resolution will be given). One linearizes the dynamical system around F_0:

$$\begin{aligned} \frac{\partial(0 + \delta Z)}{\partial \tau} &= 0 + \delta v, \\ \frac{\partial(0 + \delta v)}{\partial \tau} &= \mu(0 + \delta Z) - (0 + \delta Z)^3 \end{aligned} \tag{8}$$

that is:

$$\begin{aligned} \frac{\partial(\delta Z)}{\partial \tau} &= \delta v, \\ \frac{\partial(\delta v)}{\partial \tau} &\simeq \mu \delta Z, \end{aligned} \tag{9}$$

$$\frac{\partial}{\partial \tau}\begin{pmatrix} \delta Z \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}\begin{pmatrix} \delta Z \\ \delta v \end{pmatrix}. \tag{10}$$

The eigenvalues of the 2×2 matrix are $\lambda = \pm\sqrt{\mu}$ and the eigenvectors write:

$$v_1 = \begin{pmatrix} 1 \\ \sqrt{\mu} \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -\sqrt{\mu} \end{pmatrix}. \tag{11}$$

Hence, F_0 is a saddle node-point (homoclinic point) since it features a stable and an unstable manifolds which are connected by a closed orbit. Since the centrifugal term is non-linear, the eigenvalues and eigenvectors do not change if we include it in the equation of motion as we perform a linearization. Along the eigenvectors, the velocity $v = Z_\tau$ is proportional to the displacement Z. Hence, the behavior is exponential as $Z \simeq e^{\mp\sqrt{\mu}\tau}$ which is reminiscent of a soliton or a front: the time needed to reach or to depart from the origin is logarithmically slow. F_1 and F_2 are called centers. Indeed, it is easy to show that, around them, the equation of motion approximates to the one of an ellipse in the phase-space.

If the particle is launched in one of the two wells of the Mexican hat, the trajectories correspond to the Fig. 7(a) of the numerical simulations reported by Ochoa and Clavijo. The particle oscillates in the well around the center. If the initial amplitude is close to the center F_1 (F_2), the shape is almost sinusoidal (the trajectory in the phase-space is almost circular: see A in Fig. 7(e) in [1]) as the well is approximately parabolic (Fig. 7(a) in [1]). If the initial amplitude increases, non-linearities will affect the shape of the oscillations and one obtains the so-called "cnoidal" waves whose crests are more pronounced and whose troughs are more flat, as one can see in Fig. 7(b) in [1]. If the initial amplitude is sufficiently big, the particle moves alternatively from one well to the other, as pictured in

Fig. 7(c) of [1]. The curve in the phase-space which passes across F_0 and which encircles both F_1 and F_2 is called the separatrix (between B and C in Fig. 7(e) of [1]). For bigger initial amplitudes, the particle follows the trajectory represented in Fig. 7(d) of [1].

Now, we shall display a more exact solution for the separatrix. The solution associated with the separatrix is such that $Z(\tau)$ is not identically null and such that $Z(\pm\infty)=0$ and $\partial Z/\partial\tau(\pm\infty)=0$. Hence, $Z(\tau)$ has a bell shape with a maximum and vanishes towards $\pm\infty$. Let us return to the equation of motion:

$$Z_{\tau\tau}=\mu Z-Z^3. \tag{12}$$

One makes the following changes of variables: $\tau=\gamma X \Rightarrow X=\gamma^{-1}\tau$. We get: $\partial_\tau=\partial_X(\partial X/\partial\tau)=\gamma^{-1}\partial_X$ et $\partial_{\tau^2}=\gamma^{-2}\partial_{X^2}$. Hence:

$$Z_{XX}+\gamma^2 Z^3-\mu\gamma^2 Z=0. \tag{13}$$

With $\mu\gamma^2=1\Rightarrow\gamma=1/\sqrt{\mu}$, we have:

$$Z_{XX}+\gamma^2 Z^3-Z=0. \tag{14}$$

Let us introduce S such that $Z=\lambda S$:

$$S_{XX}+\lambda^2\gamma^2 S^3-S=0. \tag{15}$$

As a consequence, $\lambda\gamma=1\Rightarrow\lambda=1/\gamma=\sqrt{\mu}$. We obtain a dimensionless equation of motion:

$$S_{XX}+S^3-S=0 \tag{16}$$

with $\tau=X/\sqrt{\mu}$ and $Z=\sqrt{\mu}S$.

One introduces the new potential $\mathcal{V}=S^4/4-S^2/2$:

$$S_{XX}+\frac{\partial\mathcal{V}}{\partial S}=0. \tag{17}$$

One multiplies by S_X:

$$\frac{\partial}{\partial X}\left(\frac{S_X^2}{2}+\mathcal{V}(S)\right)=0 \tag{18}$$

that is:

$$\frac{S_X^2}{2}+\mathcal{V}(S)=constant. \tag{19}$$

With $X\to\infty$, $S\to 0$, $S_X\to 0$ and $\mathcal{V}\to 0$ for the separatrix, the *constant* is null. So we get:

$$\frac{S_X^2}{2}+\mathcal{V}(S)=0. \tag{20}$$

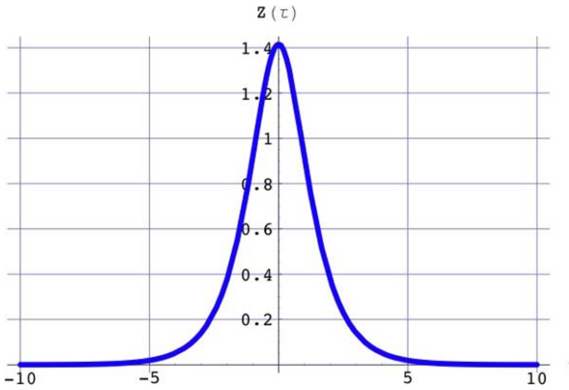

Fig. 3 Separatrix solution $Z(\tau)=\sqrt{2\mu}/\cosh(\sqrt{\mu}\tau)$ of the approximate equation for small amplitudes and without the centrifugal term with $\mu=+1$

The dimensionless Newton equation becomes:

$$\frac{S_X^2}{2}=\frac{S^2}{2}-\frac{S^4}{4} \tag{21}$$

that we rewrite in the form:

$$S_X^2=S^2\left(1-\frac{S^2}{2}\right) \tag{22}$$

that is:

$$\frac{dS}{dX}=\pm S\sqrt{1-\frac{S^2}{2}}. \tag{23}$$

One inverts it:

$$dX=\pm\frac{dS}{S\sqrt{1-\frac{S^2}{2}}}. \tag{24}$$

With a new variable $T=S/\sqrt{2}$, and hence $dS=\sqrt{2}\,dT$, one obtains:

$$dX=\pm\frac{dT}{T\sqrt{1-T^2}}. \tag{25}$$

We make the important assumption that $T\in[0,1]$ in order to ease the calculations. Of course, one can change these conditions. We use:

$$\int\frac{dT}{T\sqrt{1-T^2}}=-\cosh^{-1}\left(\frac{1}{T}\right)$$

in order to integrate the equation in T:

$$X+D=\mp\cosh^{-1}\left(\frac{1}{T}\right) \tag{26}$$

where D is a constant of integration. The sign $\mp$ corresponds to the two branches of the symmetric homocline. One considers the positive branch:

$$T(X) = \frac{1}{\cosh(X+D)}. \tag{27}$$

If $T(0) = 1$ then $D = 0$. Moreover, $Z(X) = \sqrt{\mu}S(X) = \sqrt{2\mu}T(X) = \sqrt{2\mu}T(\sqrt{\mu}\tau)$:

$$Z(\tau) = \frac{\sqrt{2\mu}}{\cosh(\sqrt{\mu}\tau)} \tag{28}$$

One recalls that $Z(0) = Z_0 = \sqrt{2\mu}$. As a consequence, the solution corresponding to the separatrix (complete solution B in Fig. 7(e) of [1] round F_0) is:

$$Z(\tau) = \frac{Z_0}{\cosh(\frac{\sqrt{2}}{2}Z_0\tau)}. \tag{29}$$

If one plots $Z(\tau)$ (see Fig. 7(b) in [1] with several bumps in a shape similar to the unique bump corresponding to the separatrix), we find a bell shape (with a maximum and a width of the order of $1/\sqrt{\mu}$), which vanishes at $\pm\infty$ with an exponential behavior $Z \simeq e^{\mp\sqrt{\mu}\tau}$, in accordance with the linear theory as discussed above, when we evaluated the eigenvectors and eigenvalues (see our Fig. 3).

However, as the geometry implies that $|Z| \le 1$, some parts of the phase-space are unreachable. More precisely, one can easily show that (provided $x_0 + r_b \ll R$):

$$\begin{aligned} -1 < -1 + \frac{(x_0+r_b)^2}{2R^2} &\le Z \\ &\le 1 - \frac{(x_0+r_b)^2}{2R^2} < 1 \end{aligned} \tag{30}$$

with r_b the radius of a bead. Hence, the existence of the separatrix solution depends on the physical parameters. Experimental evidence is needed here.

An analogy with the motion of a particle in a potential well can be used to study the case $\omega > \omega_c$. We will leave as an exercise for the reader the opportunity to show that the potential shape is almost parabolic with only one minimum (see Fig. 2). The trajectories in the phase-space are circles for small initial amplitudes and become elliptic for bigger ones, which is the typical behavior expected for a non-linear oscillator (see Fig. 8(b) in [1]). We underline that we only solved an approximation of the equation derived in [1], which is valid for small amplitudes and where the centrifugal term has been neglected.

3 Phase-space analysis via numerical simulations

In our previous comment on the BHS problem [6], we have solved approximately the equation of motion derived by Ochoa and Clavijo for the case $\omega > \omega_c$, taking into account the centrifugal term. The first (third) part of the present work was (will be) dedicated to an approximate (exact) solution neglecting (including) the centrifugal effect of the same equation when $\omega < \omega_c$. Now, we carry out a phase-space analysis of the BHS problem for both cases, $\omega >$ and $< \omega_c$. In particular, we will underline the role of the centrifugal force.

In Fig. 4, which we plot with an applet from the public domain, in order to illustrate the influence of the centrifugal effect[1] for different values of the constants α, β and γ, we show the phase-spaces corresponding to the following approximate generalized dynamical system:

$$\begin{aligned} \frac{\partial x}{\partial t} &= y, \\ \frac{\partial y}{\partial t} &= \alpha x + \beta x^3 + \gamma x y^2. \end{aligned} \tag{31}$$

We remark that the centrifugal term modifies the phase-space in the vicinity of small positions x where high velocities $\dot{x} = y$ are observed. More precisely, the effect is more pronounced for the trajectories outside the separatrix for both cases $\omega >$ and $< \omega_c$.

Now, we focus only on the case $\omega < \omega_c$. The exact generalized dynamical system writes:

$$\begin{aligned} \frac{\partial x}{\partial t} &= y, \\ \frac{\partial y}{\partial t} &= e\left(x - x^3\right) + f\frac{xy^2}{1-x^2} + gx\sqrt{1-x^2}. \end{aligned} \tag{32}$$

We recall that the range of x is limited to ± 1 due to the geometry of the BHS problem.

Clearly, two regimes appear depending on the value of $f < 0$ which is associated to the centrifugal effect. For small f (Fig. 5 top left), no oscillations are observed within the separatrix. All the trajectories end

[1] See http://www.math.rutgers.edu/courses/ODE/sherod/phase-local.html.

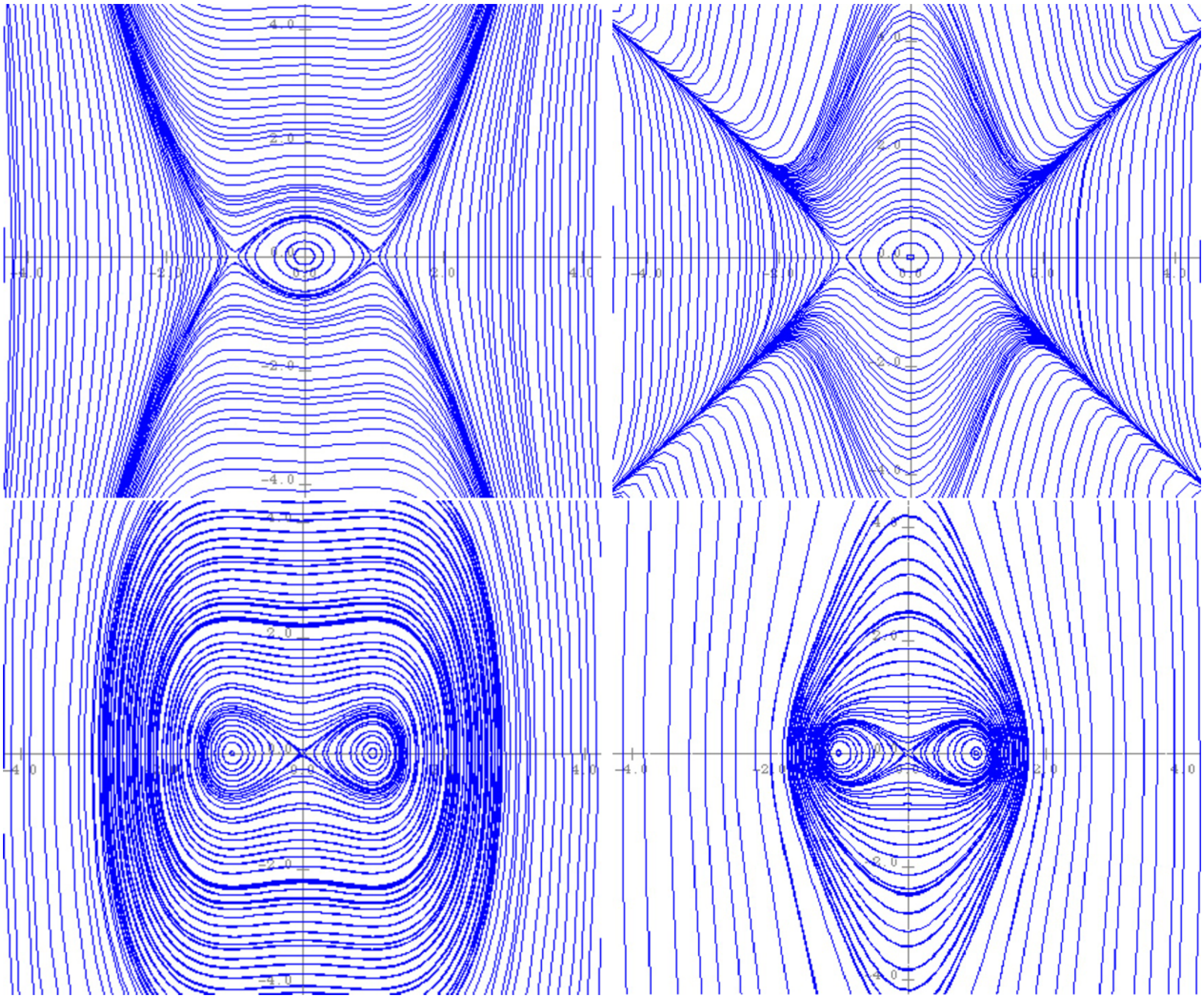

Fig. 4 Phase spaces of the approximate generalized dynamical system ($-4 \le x, y \le 4$): $[\alpha, \beta, \gamma] =$ (*top left*) $[-1, +1, 0]$; (*top right*) $[-1, +1, -1]$; (*bottom left*) $[+1, -1, 0]$ and (*bottom right*) $[+1, -1, -1]$

up within ± 1. If $|f|$ increases, basins of oscillation appear (Fig. 5 top right). For higher values, the centers get closer to the origin of the phase-space (Fig. 5 bottom left). Moreover, when $e > 0$ is increased with respect to $|f|$, the excursions of oscillations become more pronounced with high velocities (Fig. 5 bottom right).

One will show how to interpret this behavior due to an exact resolution of the BHS problem.

4 Complete case

How can we understand the physical meaning of all the terms associated with the complete equation of motion? It is enough to consider a simple pendulum. Indeed, if one denotes by θ the angle with respect to the vertical of an oscillating pendulum, one obviously has:

$$\ddot{\theta} + \sin\theta = 0. \tag{33}$$

With $X = \sin\theta$, we get:

$$\ddot{X} + \left(\frac{X}{1 - X^2}\right)\dot{X}^2 + X\sqrt{1 - X^2} = 0. \tag{34}$$

We end up with:

$$\ddot{X} + X\dot{X}^2 + X - \frac{1}{2}X^3 = 0 \tag{35}$$

if we take $X \ll 1$.

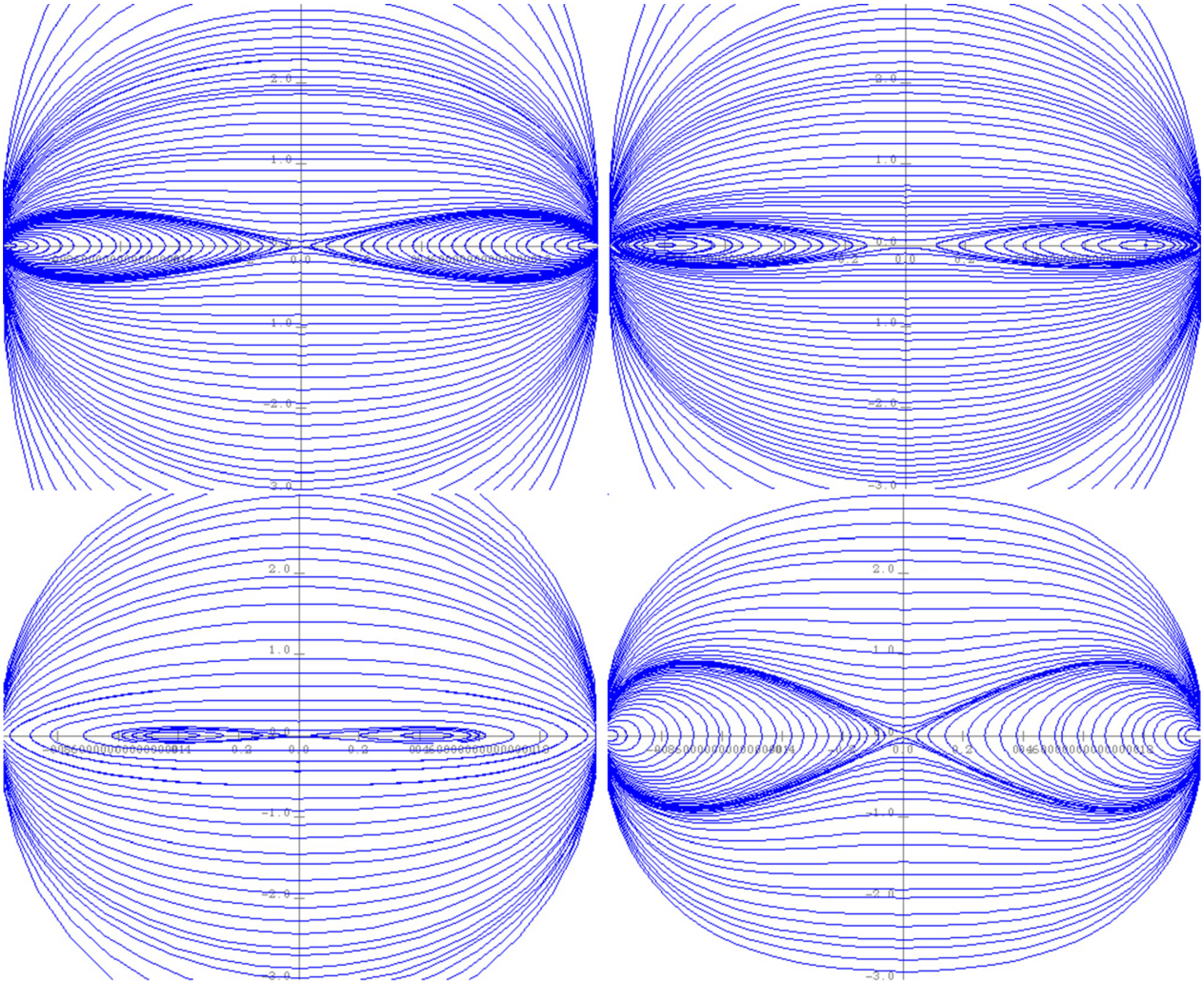

Fig. 5 Phase spaces of the exact generalized dynamical system ($-1 \le x \le 1$ & $-3 \le y \le 3$): $[e, f, g] =$ (*top left*) $[+1, -0.2, -1]$; (*top right*) $[+1, -0.6, -1]$; (*bottom left*) $[+1, -0.9, -1]$ and (*bottom right*) $[+4, -0.6, -1]$

Let us return to the full equation of evolution in the case $\omega < \omega_c$:

$$\ddot{Z} + \left(\frac{Z}{1-Z^2}\right)\dot{Z}^2 + \Omega_1^2 Z\sqrt{1-Z^2} - \Omega_0^2\big(Z - Z^3\big) = 0. \quad (36)$$

We introduce the change of a variable, $u(Z) = \dot{Z}^2$, which implies $2\ddot{Z} = du/dZ$. The initial equation becomes:

$$\frac{1}{2}\frac{du}{dZ} + \left(\frac{Z}{1-Z^2}\right)u = \Omega_0^2\big(Z - Z^3\big) - \Omega_1^2 Z\sqrt{1-Z^2}. \quad (37)$$

By solving the homogeneous equation $du/dZ + 2Zu/(1-Z^2) = 0$, we find $u = K(1-Z^2)$. Then, the constant-variation method leads to $K(Z) = \Omega_0^2 Z^2 + 2\Omega_1^2\sqrt{1-Z^2} + C$, where C is a constant. We end up with an equation of conservation:

$$\dot{Z}^2 = \Omega_0^2 Z^2\big(1-Z^2\big) + 2\Omega_1^2\big(1-Z^2\big)^{3/2} + C\big(1-Z^2\big) \quad (38)$$

that is:

$$Y(X) = (1-X)\big[\Omega_0^2 X + 2\Omega_1^2\sqrt{1-X} + C\big] = (1-X)F(X) \quad (39)$$

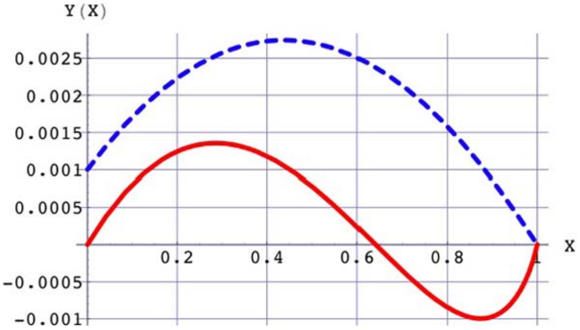

Fig. 6 $Y(X) = (1-X)[\Omega_0^2 X + 2\Omega_1^2\sqrt{1-X} + C]$: *dashed line* with $\Omega_0^2 = +0.01$, $\Omega_1^2 = +0.001$, $C = -0.001$ (no oscillations); *full line* with $\Omega_0^2 = +0.05$, $\Omega_1^2 = +0.04$, $C = -0.08$ (oscillations)

with $Y = \dot{Z}^2$ and $X = Z^2$. Remark that $(1-X)$ is always positive since $|Z| \le 1$. The different values of C correspond to several trajectories in the phase-space.

Whatever the value of C, $Z = \pm 1$ is a singularity and this implies the non-uniqueness of the solution in this problem, since several trajectories in the phase-space with different initial conditions end up within ± 1.

We discuss now the sign of $F(X)$ with the constraints $0 \le X \le 1$ and $0 \le Y$. One can show easily that $F(X)$ has a maximum $0 < X_{max} = 1 - (\Omega_1/\Omega_0)^4 < 1$ and $F(X_{max}) = C + \Omega_0^2 + \Omega_1^4/\Omega_0^2$ (see Fig. 6). Moreover, we find that $F(0) = C + 2\Omega_1^2$ and $F(1) = C + \Omega_0^2$.

The separatrix is defined by $F(0) = 0$. Now, with $C = -2\Omega_1^2$, one can linearize the previous equation in order to obtain $Y \simeq (\Omega_0^2 - \Omega_1^2)X$, that is, $\dot{Z} \simeq \pm\sqrt{\Omega_0^2 - \Omega_1^2}Z$. We do recover the exponential behavior of the simplified case near the origin of the phase-space for the separatrix.

If $-2\Omega_1^2 \le C$, all the trajectories reach the singularity ± 1. Otherwise, oscillations around the centers F_1 and F_2 are still possible if $F(0) < 0$ and $F(0) < F(X_{max})$, provided the existence of a value $X_0 < 1$ such that $F(X_0)$ vanishes (see Fig. 6). Finally, the conditions for oscillations write:

$$C < -\Omega_0^2 \quad \text{and} \quad -\Omega_1^2 - \frac{\Omega_1^4}{\Omega_0^2} < C < -2\Omega_1^2. \tag{40}$$

Now, we focus on the behavior of the trajectory in the phase-space close to the singularity $Z = \pm 1$. One introduces a negative perturbation W such that $Z = 1 + W$. Obviously, $1 - Z^2 \simeq -2W$. Hence:

$$\dot{W}^2 \simeq \left(\Omega_0^2 + C\right)(-2W) + 2\Omega_1^2(-W)^{3/2}. \tag{41}$$

As W is negative and $\dot{W}^2$ positive, one must have $-C \le \Omega_0^2$. As a consequence, the separatrix as well as the trajectories above it reach the singularity with the following behavior: $\dot{W} \sim \sqrt{-W}$. They are tangent to the vertical axis in the phase-space and come from the positive $\dot{W}$ and reach the singularity in a finite time t_f, whatever the inital conditions t_0 and W_0. Indeed, the following integral is convergent:

$$\int_{W_0}^{0} \frac{dW}{\sqrt{-2(\Omega_0^2 + C)W}} = \int_{t_0}^{t_f} dt. \tag{42}$$

One finds with $W_0 < 0$:

$$t_f = t_0 + \frac{-\sqrt{2}W_0}{\sqrt{-2(\Omega_0^2 + C)W_0}}. \tag{43}$$

5 Conclusions

In this work and in our previous comment [6], a complete resolution of the BHS problem was given. The BHS dynamical system is a nice example of the influence of centrifugal forces on the motion of a mechanical setup and can serve as a model for other problems involving inertial effects. An experimental test of the oscillation criterion which we derived would be welcome. Finally, the BHS setup features singularities in a finite time and can serve as a simple mechanical test system for more complex problems where singularities appear.

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