

The gauge non-invariance of Classical Electromagnetism

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Physical theories of fundamental significance tend to be gauge theories. These are theories in which the physical system being dealt with is described by more variables than there are physically independent degrees of freedom. The physically meaningful degrees of freedom then reemerge as being those invariant under a transformation connecting the variables (gauge transformation). Thus, one introduces extra variables to make the description more transparent and brings in at the same time a gauge symmetry to extract the physically relevant content. It is a remarkable occurrence that the road to progress has invariably been towards enlarging the number of variables and introducing a more powerful symmetry rather than conversely aiming at reducing the number of variables and eliminating the symmetry [1]. We claim that the potentials of Classical Electromagnetism are not indetermined with respect to the so-called gauge transformations. Indeed, these transformations raise paradoxes that imply their rejection. Nevertheless, the potentials are still indetermined up to a constant.

1 Introduction

In Classical electromagnetism, the electric field \mathbf{E} and the magnetic field \mathbf{B} are related to the scalar V and vector \mathbf{A} potentials by the following definitions [2] :

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (1)$$

One century ago, H.A. Lorentz noticed that the electromagnetic field remains invariant ($\mathbf{E}' = \mathbf{E}$ and $\mathbf{B}' = \mathbf{B}$) under the so-called gauge transformations [3]:

$$\mathbf{A}' = \mathbf{A} + \nabla f \quad \text{and} \quad V' = V - \frac{\partial f}{\partial t} \quad (2)$$

where $f(x, t)$ is the gauge function.

Hence, this indeterminacy is believed to be an essential symmetry of Classical Electromagnetism [3]. Moreover, it is often related to the assertion that the potentials are not measurable quantities contrary to the fields. Hence, one must specify what is called a gauge condition, that is a supplementary equation which is injected in the Maxwell equations expressed in function of the electromagnetic potentials in order to suppress this indeterminacy. It is common to say that these gauge conditions are mathematical conveniences that lead to the same determination of the electromagnetic field. In this context, the choice of a specific gauge condition is motivated from the easiness in calculations compared to another one. In a certain manner, although their mathematical expressions are different, it is supposed that they are equivalent as the fields are invariant with respect to the gauge transformations. Furthermore, no physical meaning is ascribed to the gauge conditions as the potentials are assumed not to have one... Despite these assertions which are shared by a large majority of physicists, a definition for the potentials dating back to Maxwell was recalled recently and which resolves, according to our point of view, the question of indeterminacy by giving them a physical interpretation. Moreover, we showed that the Coulomb and Lorenz gauge conditions were, in fact, not equivalent because they must be interpreted as physical constraints that is electromagnetic continuity equations [4, 5]. In addition, we were able to demonstrate that the Coulomb gauge condition is the galilean approximation of the Lorenz gauge condition within the magnetic limit of Lévy-Leblond & Le Bellac [8, 7, 6]. So, to "make a gauge choice" that is choosing a gauge condition is, as a consequence of our findings, not related to the fact of fixing a special couple of potentials. Gauge conditions are completely uncorrelated to the supposed indeterminacy of the potentials. Hence, we proposed to rename "gauge condition" by "constraint" [4, 5, 6].

In this article, we would like to reexamine the common belief concerning the assumed indeterminacy of the potentials with the assumption that the "constraint" do not fix the value of the potentials. Indeed, we will show that gauge transformations introduce paradoxes which imply their rejection. This point of view was expressed already by the school of De Broglie : by the master himself [9] or by his followers like Costa

De Beaugard [10]...

2 The case of a stationary electric field

Imagine a one dimensional **stationary** problem defined by the following potentials :

$$\mathbf{A} = \mathbf{0} \quad \text{and} \quad V(x) = -Ex \quad (3)$$

One finds easily :

$$\mathbf{B} = \mathbf{0} \quad \text{and} \quad \mathbf{E} = E\mathbf{e}_x \quad (4)$$

The electric and magnetic fields are constants in time.

Now, we can perform a gauge transformation with this particular gauge function :

$$f(x, t) = -Ext \quad (5)$$

The new potentials are :

$$\mathbf{A}' = -\mathbf{E}t \quad \text{and} \quad V' = V - \frac{\partial f}{\partial t} = 0 \quad (6)$$

Of course, the electric field is unchanged but is the underlying physics expressed by the potentials the same ? We believe that an electric field can be created by two very different physical processes that is time variation of a vector potential (like in induction phenomena) or space variation of a scalar potential (like in the electron gun). We are in front of the first paradox : how can a physical quantity (here, the vector potential) be a function of time in a stationary problem ? In the case of a capacitor for example, the static electric field is created by static electric charges on the plates of the capacitor. If one admits that one can describe this static electric field by a time-dependent vector potential, one must admit that the sources (charges or currents) of this electric field are time-dependent which is not experimentally the case.

The unconvinced reader could argue that we proposed a gauge function which depends explicitly on time in a stationary problem. As a matter of fact, if $f(x, t)$ does not depend on time, the scalar potential is invariant $V' = V$ and so is not indetermined with respect to the gauge transformations.

In a stationary problem, the electric field is expressed as $\mathbf{E} = -\nabla V$ whereas the magnetic field is still defined as previously. In this case, the vector potential could still be indetermined. So, are two vector potentials differing from a gradient physically equivalent ?

3 The case of a vector potential equal to a gradient

Now, one will apply the so-called Stokes-Helmholtz-Hodge decomposition to the vector potential :

$$\mathbf{A} = \mathbf{A}_{\text{longitudinal}} + \mathbf{A}_{\text{transverse}} \quad (7)$$

with :

$$\mathbf{A} = \nabla g + \nabla \times \mathbf{R} \quad (8)$$

where g is a scalar and \mathbf{R} a vector. The decomposition is unique up to the additive gradient of a harmonic function with the following properties [11]:

$$\nabla \cdot \mathbf{A}_{\text{harmonic}} = 0 \quad \nabla \times \mathbf{A}_{\text{harmonic}} = 0 \quad (9)$$

If we use gauge transformations, we can notice that only the longitudinal (and/or harmonic) part of the vector potential and the scalar potential are affected by these transformations. The transverse part remains unchanged. Moreover, the magnetic field depends only on the transverse part. So, if there is indeterminacy, it must imply indeterminacy of the longitudinal (and/or harmonic) part. As a consequence, the longitudinal (and/or harmonic) part cannot have a physical meaning if it is indetermined with respect to the gauge transformations.

Usually, the vector potential is equal to its transverse part in most of the problems of Classical Electromagnetism. For example, the vector potential for a magnet is expressed by :

$$\mathbf{A} = \mathbf{A}_{\text{transverse}} = \frac{\mu_0}{4\pi} \nabla \times \left(\frac{\mathbf{m}}{r} \right) \quad (10)$$

where \mathbf{m} is the strength of the poles (the so-called magnetic mass or moment).

In this case, we observe a magnetic field by definition. And, if the vector potential varies in time, it creates an electric field again by definition : the time integral of the electric field could be considered as a direct measure of the vector potential without the presence of static charges that is of a scalar potential. If not, one can use the superposition

theorem to evaluate first the part of the electric field associated to the static charge (that is the scalar potential) and then the part associated to the current (that is the vector potential) if and only if the separation is possible...

At this stage, the question is to know whether a vector potential only equal to a gradient can have a physical effect when the electromagnetic field is null.

Outside a solenoid, the vector potential is precisely equal to a gradient as expressed by the following formula in cylindrical coordinates (r, θ) [2] :

$$\mathbf{A} = \frac{\Phi}{2\pi r} \mathbf{e}_\theta = \frac{\Phi \nabla \theta}{2\pi} \quad (11)$$

where Φ is the flux of magnetic field inside the solenoid or the circulation of the vector potential outside the solenoid. More precisely, its curl and divergence are null so the vector potential outside a solenoid is of a harmonic-type according to the indeterminacy of the Stokes-Helmholtz-Hodge decomposition [11]. Moreover, we point out forcefully that the mathematical indeterminacy due to the gauge transformations is discarded by the boundary conditions which give a physical determination to the vector potential outside a solenoid : the vector potential vanishes far from its current sources. The external region of a solenoid is one of the numerous experimental configurations to observe the well-known Aharonov-Bohm effect [12]. So, we are in front of the second paradox : the Aharonov-Bohm effect contradicts the fact that a vector potential equal only to a gradient could not have a physical effect.

However, one usually argues that the Aharonov-Bohm effect is a quantum effect and that the potentials can have a meaning in quantum physics and not in classical physics. Moreover, the prediction of the Aharonov-Bohm effect pointed out an unanticipated way in which the vector potential can affect a measurement on a charged particle in a region of zero field, but the predicted phase is the result of a path integral which insures that the result is gauge independent : it is not the result of a local (gauge-dependent) value of the vector potential.

Why add in the path integral a (mathematical) gradient to the vector potential which is already a (physical) gradient ?

A vector potential equal only to a gradient should not have a physical effect as imply by gauge transformations because we could cancel the longitudinal (and/or harmonic) part by adding the gradient of the appropriate gauge function.

In the solenoid example, we can take as a gauge function $f(r, \theta, t) = -\Phi\theta/2\pi$ and still there is experimentally an effect despite the fact that all the potentials and so the fields cancel outside the solenoid.

However, one can find in the literature some theoretical arguments against this gauge transformation according to the fact that the existence of the solenoid implies that the space is not simply-connected. Yet, it is true that in a multiply connected region, the function α which characterises the longitudinal (and/or harmonic) part of the vector potential in the general case becomes multivalued but the longitudinal (and/or harmonic) part of the vector potential (the only one which is different from zero outside the solenoid) is not multivalued as one takes the gradient of α .

Another argument is to remember that the gauge transformations were introduced by Lorentz without any constraint on the connectness of the space.

Having the Aharonov-Bohm effect in mind, we can recall now a very simple experiment which cannot be explained with Maxwell equations expressed in function of the electromagnetic field only and which shows the physical character of a longitudinal (and/or harmonic) vector potential in classical physics.

Let's take again the geometry of the solenoid. If the current varies with time the magnetic field is still null outside the solenoid but because the vector potential is not null outside the solenoid and varies with time, it creates an electric field outside the solenoid. If we denote the flux of the magnetic field inside the solenoid (or the circulation of the vector potential outside the solenoid) $\Phi = LI$ where L is the inductance of the solenoid and I the current intensity, the electric field is expressed by :

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{L}{2\pi r} \frac{dI}{dt} \mathbf{e}_\theta \quad (12)$$

If we apply Maxwell equations expressed in function of the fields with the prescription that the magnetic field is null outside the solenoid, we only find that the electric field is lamellar outside the solenoid which is supposed to be infinite ($\nabla \times \mathbf{E} = 0$ because $\partial \mathbf{B} / \partial t = \mathbf{0}$ even in this time-dependent problem because $\mathbf{B} = \mathbf{0}$ outside the solenoid)...

This experiment is carried out very easily. It demonstrates that a vector potential only equal to a gradient can have a physical effect in Classical Electromagnetism when it varies in time and thus creates by

definition an electric field. Of course, for a finite solenoid, the leaking magnetic field is not null outside the solenoid. However, it creates a leaking electric field which is negligible and opposite with respect to the electric field created by the contribution of the vector potential due to the ideal solenoid...

Another example of a physical vector potential which is equal to a gradient appears in the well-known Meissner effect in supraconductivity and it was discussed nicely by Tonomura [13].

4 The case of a uniform magnetic field

Another drawback of the gauge transformations can be illustrated by the following example : one often finds in textbooks that we can describe a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ by either the so-called symmetric "gauge" $\mathbf{A}_s = \mathbf{1}/2\mathbf{B} \times \mathbf{r}$ or by the so-called Landau "gauge" [3]. This two "gauges" are related by a gauge transformation :

$$\mathbf{A}_1 = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{1}{2}[-By, Bx, 0] \quad (13)$$

becomes either :

$$\mathbf{A}_2 = [0, Bx, 0] \quad \text{or} \quad \mathbf{A}_3 = [-By, 0, 0] \quad (14)$$

with the gauge functions $\pm f = \pm xy/2$.

However, there is no discussion in the litterature of the following issue. As a matter of fact, if we consider a solenoid with a current along \mathbf{e}_θ , the magnetic field is uniform (along \mathbf{e}_z) and could be described by the symmetric "gauge" or the Landau "gauge". Yet, the vector potential in the Landau "gauge" \mathbf{A}_2 is along \mathbf{e}_y whereas the vector potential in the symmetric gauge is along \mathbf{e}_θ . We advocate that only the symmetric "gauge" is valid in this case because it does respect the symmetry of the currents ($\mathbf{J} = J\mathbf{e}_\theta$) whereas the Landau "gauge" does not. Moreover, the symmetric "gauge" (or the Landau "gauge") is not, in fact, a gauge condition but a solution describing a uniform magnetic field under the Coulomb constraint ($\nabla \cdot \mathbf{A}_1 = \mathbf{0}$). In order to understand this last point, one can picture an analogy between Fluid Mechanics and Classical Electromagnetism. Indeed, the solenoid is analogous to a cylindrical vortex core with vorticity \mathbf{w} and we know that the velocity inside the core is given by $\mathbf{u} = \mathbf{1}/2\mathbf{w} \times \mathbf{r}$ which is analogous to the symmetric gauge for

an incompressible flow ($\nabla \cdot \mathbf{u} = 0$). Outside the vortex core, the velocity is given by [14] :

$$\mathbf{u} = \frac{\Gamma \nabla \theta}{2\pi} = \frac{\Gamma}{2\pi r} \mathbf{e}_\theta \quad (15)$$

where Γ is the flux of vorticity inside the vortex or the circulation of the velocity outside the vortex. One recovers the analogue formula for the vector potential outside a solenoid...

Of course, if the problem we are considering does not feature the cylindrical geometry (two horizontal plates with opposite surface currents for example, analogous to a plane Couette flow [14]), one of the Landau gauges \mathbf{A}_2 or \mathbf{A}_3 must be used instead of the symmetric gauge \mathbf{A}_1 according to the necessity of respecting the underlying distribution/symmetry of the currents which is at the origin of both the vector potential and the magnetic field. To give a magnetic vector field without specifying its current source is an ill-posed problem which was interpreted so far by attributing an indeterminacy to the vector potential which is wrong.

Now, how can we test experimentally this argument based on symmetry ? If the current of the solenoid varies with time, it will create an electric field which is along \mathbf{e}_θ as the vector potential because the electric field is minus the time derivative of the vector potential. If the currents in the horizontal plates change with time, a horizontal electric field will appear for the same reason. The author rejects all the arguments based on the fact that one can define through a gauge transformations a time dependent scalar potential which would explain the observed electric field. Indeed, a scalar potential is physically defined with respect to charge distributions and not current distributions.

Contrary to the common belief, it is possible to discriminate experimentally between two vector potentials (related by a gauge transformation) creating a uniform magnetic field. We have shown that the symmetry of the current source implies a certain distribution of the vector potential which is at the origin of an electric field when the intensity is time-dependent. Its orientation is dictated by the vector potential alone which, eventhough is not observable by itself, has observable consequences.

5 Conclusions

What is the meaning of gauge transformations ? We believe that it is only a structural feature (that is linearity) of the definitions of the po-

tentials from the fields. The potentials of Classical Electromagnetism do have a physical meaning as recalled recently [4, 5, 6, 15, 16] and should be considered as the starting point of Classical Electromagnetism [6, 15]. If we defined the fields from the potentials and not the contrary, the gauge transformations loose their sense because they imply the paradoxes raised in this article. As a conclusion, we must reject gauge transformations. Gauge invariance is preserved but in a weaker sense : the potentials are defined up to a constant. This constant is equal to zero when the sources are confined to a certain region of space : one assumes that the potentials vanish at infinity far from their sources. If the domain is bounded like in a Faraday cage, the surface potentials are given by the contribution of all the sources outside the region of interest [17] : one makes the assumption that their knowledge is not important as one measures only differences of potentials according to their definition in function of a constant of reference [6]. We recall for example that the vector potential is an electromagnetic impulsion that is a difference of electromagnetic momentum. In mechanics, a momentum is indetermined as it is defined with respect to a reference but the impulsion constructed from this momentum has a definite value so is not indetermined...

References

- [1] Henneaux M. & Teitelboim C., Quantization of gauge systems, Princeton University Press, Princeton, 1992.
- [2] Jackson J.D., Classical Electrodynamics, third edition, John Wiley & Sons, Inc., 1998.
- [3] Jackson J.D. & Okun L.B., Historical roots of gauge invariance, Reviews of Modern Physics, Vol. 73, p. 663-680, 2001.
- [4] Rousseaux G. & Guyon E., A propos d'une analogie entre la mécanique des fluides et l'électromagnétisme, Bulletin de l'Union des Physiciens, **841** (2), p. 107-136, février, 2002.
(Article available online at :
<http://www.udppc.asso.fr/bup/841/0841D107.pdf>).
- [5] Rousseaux G., On the physical meaning of the gauge conditions of Classical Electromagnetism : the hydrodynamics analogue viewpoint, Annales de la Fondation Louis de Broglie, Volume 28, Numéro 2, p. 261-270, 2003.
(Article available online at :
<http://www.ensmp.fr/aflb/AFLB-282/aflb282p261.pdf>)
- [6] Rousseaux G., Sur la théorie de Riemann-Lorenz de l'Électromagnétisme Classique, Bulletin de l'Union des Professeurs de Physique et de Chimie, Vol. 98, 868 (2), p. 41-56, Novembre 2004.

- [7] Rousseaux G. & Doms A., Remarques supplémentaires sur l'approximation des régimes quasi-stationnaires en électromagnétisme, Bulletin de l'Union des Professeurs de Physique et de Chimie, Vol. 98, 868 (2), p. 71-86, Novembre 2004.
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[http : //www.udppc.asso.fr/bup/868/0868D071.pdf](http://www.udppc.asso.fr/bup/868/0868D071.pdf))
- [8] Le Bellac M. & Lévy-Leblond J.-M., Galilean Electromagnetism, Il Nuovo Cimento, Vol. 14B, N. 2, 11 Aprile, p. 217-233, 1973.
- [9] De Broglie L., Diverses questions de mécanique et de thermodynamique classiques et relativistes, Springer, End of Chapter 3, 1995.
- [10] Costa de Beauregard O., Un énoncé de Vaschy et une expérience de Blondel revisités : la tension d'Ampère, Annales de la Fondation Louis de Broglie, Volume 28, Numéro 1, p. 77, 2003.
- [11] Dassios G. & Lindell I.V., Uniqueness and reconstruction for the anisotropic Helmholtz decomposition, Journal of Physics A : Mathematical and General, 35 (24), p. 5139-5146, 2002.
- [12] Feynman R., Leighton R. & Sands M., The Feynman Lectures on Physics, Addison Wesley, Reading, Ma., Vol. 2, p. 15-7/15-14, 1964.
- [13] Tonomura A., The quantum world unveiled by electron waves, World Scientific, 1998.
- [14] Guyon E., Hulin J.P., Petit L. & Mitescu C.D., Physical Hydrodynamics, Oxford University Press, 2001.
- [15] Mead C., Collective Electrodynamics : Quantum Foundations of Electromagnetism, M.I.T. Press, 2002.
- [16] Kholmetskii A.L., Remarks on momentum and energy flux of a non-radiating electromagnetic field, Annales de la Fondation Louis De Broglie, Volume 29, Numéro 3, p. 1-33, 2004.
- [17] Mourier G., L'invariance de jauge et la notion de système isolé : que signifie l'invariance de jauge pour un expérimentateur ?, Annales de la Fondation Louis De Broglie, Volume 26, Numéro spécial, p. 345-351, 2001.

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