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On the electrodynamics of Minkowski at low velocities

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Abstract – The Galilean constitutive equations for the electrodynamics of moving media are derived for the first time. They explain all the historic and modern experiments which were interpreted so far in a relativistic framework assuming the constant light celerity principle. Here, we show the latter to be sufficient but not necessary.

One century ago, Hermann Minkowski formulated, for the first time, a covariant theory of electrodynamics in moving media [1–4]. He generalized the studies of Henri Poincaré [5] and Albert Einstein [6] which were restricted to vacuum. The theory was not only a consequence of the relativity principle formulated by Poincaré but also of the constant light celerity principle formulated by Einstein. Hence, it should have been (and was) applied to the optics of moving media [7] and to fast particles in media [8]. Moreover, it was thought to be the only possible explanation for the entire electrodynamics of moving media especially for slow motions [9]. The story almost ended with the last experiment of electrodynamics in moving media due to the Wilsons [10] which proved the correctness of the special theory of relativity against the older theories of Hertz, Lorentz and others whose fields transformations did not match with the group additivity as implied by the relativity principle. Nowadays, only some review papers appear from time to time dealing with the electrodynamics of moving media especially on the well-known Minkowski-Abraham controversy about the correct expression for the energy-momentum tensor in media [3,4,11].

However, in 1973, Michel Le Bellac and Jean-Marc Lévy-Leblond postulated the existence of a Galilean limit of Maxwell equations that is without assuming the finiteness of the light velocity. More cumbersome, they showed that, contrary to Mechanics, Classical Electromagnetism features TWO Galilean limits: one applies to dielectrics and the other to magnets [12]. Following our recent works on Galilean Electromagnetism [13–17], here we solve the problem of the electrodynamics of moving media at low velocities. For this purpose, we derive for the first time the TWO Galilean limits of the relativistic constitutive equations introduced by Minkowski as long ago as 1908.

We first recall from the textbooks the relativistic electrodynamics of moving media before taking its Galilean limits. All the experiments of Classical Electromagnetism involving motion of part of the setup are described by this "new" theory as soon as velocities do not reach the celerity of light. It is useless to speak of applications since this theory encompasses all our wave-less technology. Needless to add that the Galilean theory is simpler than the relativistic theory...

Relativistic electrodynamics of moving media. – The relativistic form of Maxwell equation in continuous media is written as

$$\begin{aligned} \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, && \text{Faraday,} \\ \nabla \cdot \mathbf{B} &= 0, && \text{Thomson,} \\ \nabla \times \mathbf{H} &= \mathbf{j} + \partial_t \mathbf{D}, && \text{Ampere,} \\ \nabla \cdot \mathbf{D} &= \rho, && \text{Gauss,} \end{aligned} \tag{1}$$

This set is the so-called "Maxwell-Minkowski equations".

A *Poincaré-Lorentz transformation* (without rotation) acts on space-time coordinates as follows (see, for instance, Section 7.2 of [18]):

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \gamma \mathbf{v} t + (\gamma - 1)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{x})}{\mathbf{v}^2}, \\ t' &= \gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right), \end{aligned} \tag{2}$$

where $\mathbf{v}$ is the relative velocity and $\gamma = \dfrac{1}{\sqrt{1-(v/c)^2}}$.

Under a Poincaré-Lorentz-Minkoswki transformation, eq. (2), the electric and magnetic fields and their related

[(a)] E-mail: Germain.Rousseaux@unice.fr inductions transform as

$$\begin{aligned}
\mathbf{E}' &= \gamma\left(\mathbf{E} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{\mathbf{v}^2} + \mathbf{v}\times\mathbf{B}\right),\\
\mathbf{B}' &= \gamma\left(\mathbf{B} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{B})}{\mathbf{v}^2} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right),\\
\mathbf{D}' &= \gamma\left(\mathbf{D} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{D})}{\mathbf{v}^2} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H}\right),\\
\mathbf{H}' &= \gamma\left(\mathbf{H} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{H})}{\mathbf{v}^2} - \mathbf{v}\times\mathbf{D}\right),
\end{aligned} \tag{3}$$

in order to respect the covariance of eq. (1) with respect to the relativity principle [1,9].

The medium in motion is supposed to be linear, homogeneous and isotropic. ϵ (μ) denotes its permittivity (permeability). Hence, the constitutive equations in the moving frame (Minkowski's crucial hypothesis)

$$\begin{aligned}\mathbf{D}' &= \epsilon\mathbf{E}',\\ \mathbf{B}' &= \mu\mathbf{H}',\end{aligned} \tag{4}$$

become

$$\begin{aligned}
&\mathbf{D} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{D})}{\mathbf{v}^2} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H} =\\
&\quad\epsilon\left(\mathbf{E} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{\mathbf{v}^2} + \mathbf{v}\times\mathbf{B}\right),\\
&\mathbf{B} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{B})}{\mathbf{v}^2} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E} =\\
&\quad\mu\left(\mathbf{H} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{H})}{\mathbf{v}^2} - \mathbf{v}\times\mathbf{D}\right).
\end{aligned} \tag{5}$$

The scalar product with the velocity $\mathbf{v}$ of eq. (5) gives

$$\begin{aligned}\mathbf{D}\cdot\mathbf{v} &= \epsilon\mathbf{E}\cdot\mathbf{v},\\ \mathbf{B}\cdot\mathbf{v} &= \mu\mathbf{H}\cdot\mathbf{v},\end{aligned} \tag{6}$$

which allows a simplification

$$\begin{aligned}\mathbf{D} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H} &= \epsilon(\mathbf{E}+\mathbf{v}\times\mathbf{B}),\\ \mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E} &= \mu(\mathbf{H}-\mathbf{v}\times\mathbf{D}).\end{aligned} \tag{7}$$

The latter relativistic constitutive equations were written for the first time by Hermann Minkowski in his groundbreaking paper of 1908 [1].

Now, if we write $\mathbf{D}$ and $\mathbf{H}$ as a function of $\mathbf{E}$ and $\mathbf{B}$ using eq. (6) and the formula for the double vectorial product

$$\begin{aligned}\mathbf{v}\times(\mathbf{v}\times\mathbf{D}) &= \mathbf{v}(\mathbf{v}\cdot\mathbf{D}) - \mathbf{v}^2\mathbf{D},\\ \mathbf{v}\times(\mathbf{v}\times\mathbf{E}) &= \mathbf{v}(\mathbf{v}\cdot\mathbf{E}) - \mathbf{v}^2\mathbf{E},\end{aligned} \tag{8}$$

we end up with the relativist expressions for the excitations as a function of the fields in the laboratory frame

$$\begin{aligned}
\mathbf{D} &= \gamma^2\epsilon\left[\left(1-\frac{\mathbf{v}^2}{\mu\epsilon c^4}\right)\mathbf{E} + \left(1-\frac{1}{\mu\epsilon c^2}\right)\left(\mathbf{v}\times\mathbf{B} - \frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}\right)\right)\right],\\
\mathbf{H} &= \frac{\gamma^2}{\mu}\left[(1-\mu\epsilon\mathbf{v}^2)\mathbf{B} + \left(\mu\epsilon - \frac{1}{c^2}\right)(\mathbf{v}\times\mathbf{E} + \mathbf{v}(\mathbf{v}\cdot\mathbf{B}))\right].
\end{aligned} \tag{9}$$

These are the usual transformations used in the entire Physics literature.

It is a matter of simple calculations from eqs. (6) and (8) to get

$$\begin{aligned}
\mathbf{D} &= \epsilon\mathbf{E} + \gamma^2\left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times\left(\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right),\\
\mathbf{H} &= \frac{\mathbf{B}}{\mu} + \gamma^2\left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times(\mathbf{E}+\mathbf{v}\times\mathbf{B}).
\end{aligned} \tag{10}$$

As we will see shortly, these are the form of the constitutive equations more amenable to a Galilean limit.

Galilean electrodynamics of moving media. – Following the procedure adopted in ref. [13–15], a Galilean limit is obtained in two steps. First, we introduce the quasi-static approximation $v \ll c$ to get equations which do not obey the group additivity property:

$$\begin{aligned}
\mathbf{D} &\simeq \epsilon\mathbf{E} + \left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times\left(\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right),\\
\mathbf{H} &\simeq \frac{\mathbf{B}}{\mu} + \left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times(\mathbf{E}+\mathbf{v}\times\mathbf{B}).
\end{aligned} \tag{11}$$

At this stage, only the FitzGerald-Lorentz contraction factor is put to unity. For example, Landau and Lifshitz used these first-order Lorentz transformations in their textbook on electrodynamics of continuous media [19]. However, they do not form a group for the addition property and cannot be considered as correct [16].

Next, an assumption on the relative magnitude of the remaining terms must be added in order to drop the ones which break the Galilean covariance. It is easy to see that the magnetic limit corresponds to the assumption $E_m \sim vB_m \ll cB_m$ (see [13–16]). Hence, the Galilean magnetic constitutive equations write:

$$\begin{aligned}
\mathbf{D}_m &\simeq \epsilon\mathbf{E}_m + \left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times\mathbf{B}_m,\\
\mathbf{H}_m &\simeq \frac{\mathbf{B}_m}{\mu},
\end{aligned} \tag{12}$$

whereas the electric limit corresponds to $cB_e \sim vE_e/c \ll E_e$ (see [13–16]) with the following Galilean electric constitutive equations:

$$\begin{aligned}
\mathbf{D}_e &\simeq \epsilon\mathbf{E}_e,\\
\mathbf{H}_e &\simeq \frac{\mathbf{B}_e}{\mu} + \left(\epsilon - \frac{1}{\mu c^2}\right)\frac{\mathbf{v}\times\mathbf{E}_e}{c^2}.
\end{aligned} \tag{13}$$

A similar simplification could have be done with eqs. (9) instead of eqs. (10) used here. Both sets (12) and (13) do form a group. Moreover, they can be obtained directly from Minkowski constitutive equations (7) since the magnetic limit transformations follow from

$$\begin{aligned}
\mathbf{D}_m + \frac{1}{c^2}\mathbf{v}\times\mathbf{H}_m &\simeq \epsilon(\mathbf{E}_m + \mathbf{v}\times\mathbf{B}_m),\\
\mathbf{B}_m &\simeq \mu\mathbf{H}_m,
\end{aligned} \tag{14}$$

in accordance with the Galilean magnetic Maxwell-Minkowski equations

$$\begin{aligned} \nabla \times \mathbf{E}_m &= -\partial_t \mathbf{B}_m, && \text{Faraday,} \\ \nabla \cdot \mathbf{B}_m &= 0, && \text{Thomson,} \\ \nabla \times \mathbf{H}_m &= \mathbf{j}_m, && \text{Ampere,} \\ \nabla \cdot \mathbf{D}_m &= \rho_m, && \text{Gauss,} \end{aligned} \tag{15}$$

whereas the electric limit transformations come from

$$\begin{aligned} &\mathbf{D}_e \simeq \epsilon \mathbf{E}_e, \\ &\mathbf{B}_e - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}_e \simeq \mu(\mathbf{H}_e - \mathbf{v} \times \mathbf{D}_e), \end{aligned} \tag{16}$$

in accordance with the Galilean electric Maxwell-Minkowski equations

$$\begin{aligned} \nabla \times \mathbf{E}_e &= 0, && \text{Faraday,} \\ \nabla \cdot \mathbf{B}_e &= 0, && \text{Thomson,} \\ \nabla \times \mathbf{H}_e &= \mathbf{j}_e + \partial_t \mathbf{D}_e, && \text{Ampere,} \\ \nabla \cdot \mathbf{D}_e &= \rho_e, && \text{Gauss,} \end{aligned} \tag{17}$$

Le Bellac and Lévy-Leblond underlined in their seminal paper that combinations of the electric and magnetic limits are of course possible [12]. Hence, the following Galilean displacement field is allowed:

$$\mathbf{D}_G = \mathbf{D}_m + \mathbf{D}_e \simeq \epsilon(\mathbf{E}_e + \mathbf{E}_m) + \left(\epsilon - \frac{1}{\mu c^2}\right) \mathbf{v} \times \mathbf{B}_m. \tag{18}$$

The measurements by the Wilsons and their modern reproduction by Hertzberg *et al.* displayed unambiguously the factor $(\epsilon - 1/\mu c^2)$ and a linear relationship with the velocity of the displacement field [10]. However, what this experiment validates is first of all the Galilean Electrodynamics *à la* Minkowski that we derived for the first time in this work. The special relativity prediction was not so far tested contrary to what was believed and is unlikely to be because of the rapid velocities it implies. Hence, *Minkowski's electrodynamics is useless when one deals with low velocities*. However, *only the Maxwell-Minkowski equations are able to predict correctly the optics of moving media* like the Čerenkov radiation [8] or the Fresnel-Fizeau drag [7] ...

We believe to have solved the long-standing problem of the electrodynamics of moving media. For this purpose, we derived for the first time the Galilean constitutive equations for moving bodies with both dielectric and magnetic properties. Contrary to Mechanics which features only one Galilean limit due to causality, Classical Electromagnetism displays two distinct low-velocities approximations which were mixed incoherently before and generalized after special relativity was created one century ago by the joined efforts of Lorentz, Poincaré, Einstein and Minkowski.

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