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Lorenz or Coulomb in Galilean electromagnetism?

G. Rousseaux
INLN-UMR 6618 CNRS - 1361 route des Lucioles
06560 Valbonne, France

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Abstract. – Galilean electromagnetism was discovered thirty years ago by Lévy-Leblond and Le Bellac. However, these authors only explored the consequences for the fields and not for the potentials. Following De Montigny *et al.*, we show that the Coulomb gauge condition is the magnetic limit of the Lorenz gauge condition whereas the Lorenz gauge condition applies in the electric limit of Lévy-Leblond and Le Bellac. Contrary to De Montigny *et al.*, who used Galilean tensor calculus, we use orders of magnitude based on physical motivations in our derivation.

Introduction. – Does there exist a Galilean limit of Maxwell equations? According to Lévy-Leblond and Le Bellac, the answer is positive [1]. Indeed, following the work of Lévy-Leblond on Galilean invariance within Classical and Quantum Mechanics [2], they have shown that there exist not one as in mechanics but two well-defined Galilean limits of the full set of Maxwell equations: the magnetic limit and the electric limit. The two Galilean limits were introduced by Lévy-Leblond and Le Bellac without demonstration (see later for a justification). More precisely, they have shown that two particular approximations of the full set of Maxwell equations were in agreement with the two Galilean transformations for the field they "derived".

If one denotes $\gamma = (1-\boldsymbol{v}^2/c_L^2)^{-1/2}$, where c_L is the light velocity, the relativistic transformations for the fields in vacuum between two inertial frames with relative velocity $\boldsymbol{v}$ are

$$\boldsymbol{E}' = \gamma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) + \frac{(1-\gamma)(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{v}}{\boldsymbol{v}^2} \quad \text{and} \quad \boldsymbol{B}' = \gamma\big(\boldsymbol{B}-(1/c_L^2)\boldsymbol{v}\times\boldsymbol{E}\big) + \frac{(1-\gamma)(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{v}}{\boldsymbol{v}^2}. \quad (1)$$

In vacuum, one obtains the magnetic limit by stating that $|\boldsymbol{v}|/c_L \ll 1$ and $E \ll c_L B$ in (1). Conversely, the electric limit is obtained by stating that $|\boldsymbol{v}|/c_L \ll 1$ and $E \gg c_L B$. Hence, one ends up with two sets of low-velocity formulae from the Lorentz transformations [1]:

Electric limit:	Magnetic limit:
$\boldsymbol{E}' = \boldsymbol{E}$ and $\boldsymbol{B}' = \boldsymbol{B} - (1/c_L^2)\boldsymbol{v}\times\boldsymbol{E}$	$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$ and $\boldsymbol{B}' = \boldsymbol{B}$

These two limits are practically very important since they correspond to the so-called electro-quasi-static and magneto-quasi-static approximations of engineering electromagnetism as described in [3]. Moreover, magnetohydrodynamics relies on the magnetic limit whereas electrohydrodynamics relies on the electric limit of Maxwell equations [4–6].

Several authors have discussed recently Lévy-Leblond and Le Bellac paper. Holland and Brown argued that the limit process applied to the scalar and vector potential would break gauge invariance in such a way that they did not explore as Lévy-Leblond and Le Bellac the consequences for the so-called gauge conditions [7]. De Montigny *et al.* in a series of papers revisited also Galilean electromagnetism with the help of a "Galilean tensor calculus" which consists in expressing non-relativistic equations in a covariant form with a five-dimensional Riemannian manifold, the so-called Bargmann space-time approach (see [8, 9] and references therein). In their review on Galilean electromagnetism, De Montigny *et al.* have shown that the Lorenz gauge condition $\nabla \cdot \boldsymbol{A} + \frac{1}{c_L^2}\frac{\partial V}{\partial t} = 0$, which is covariant with respect to the Lorentz transformations, becomes the Coulomb gauge condition $\nabla \cdot \boldsymbol{A} = 0$ within the magnetic limit and that the Lorenz gauge condition keeps unchanged within the electric limit [8]. The present author has reached independently the same conclusions [10] by imposing directly Galilean covariance with respect to the gauge conditions depending on the Galilean transformations for the potentials, first introduced by Lévy-Leblond and Le Bellac, which differ according to the two limits [1]:

Electric limit:	Magnetic limit:
$V' = V$ and $\boldsymbol{A}' = \boldsymbol{A} - \frac{\boldsymbol{v}V}{c_L^2}$	$V' = V - \boldsymbol{v} \cdot \boldsymbol{A}$ and $\boldsymbol{A}' = \boldsymbol{A}$

by recalling that the Galilean transformations for the spatial and temporal derivations are

$$\nabla = \nabla', \tag{2}$$

$$\partial_t + \boldsymbol{v} \cdot \nabla = \partial_{t'} . \tag{3}$$

Here, we would like to show a physically meaningful derivation based on orders of magnitude of the Galilean limits for the Lorentz-covariant Lorenz gauge condition.

The Galilean limits of Lorenz gauge condition. – Now, how do Lévy-Leblond and Le Bellac know that $E \ll c_L B$ or $E \gg c_L B$? Indeed, it is a rather formal assumption which is not justified at all *a priori* whereas it is true!

We argue that the derivation of Lévy-Leblond and Le Bellac is equivalent to evaluate the order of magnitude of the following parameters:

$$\varepsilon = \frac{L}{c_L \tau} \qquad \text{and} \qquad \xi = \frac{\tilde{j}}{\tilde{\rho} c_L}\,, \tag{4}$$

where $L(\tau)$ represents the order of magnitude of a typical scale (time) of the problem and $\tilde{j}$ ($\tilde{\rho}$) represents the order of magnitude of the current (charges) density in the system under examination.

As a matter of fact, the values of the electric and magnetic fields depend on their sources, that is, on the distribution of the charge and current densities. If one evaluates the order of magnitude of the fields in function of the sources using Gauss and Ampère's equations and (4), one ends up with

$$\frac{\tilde{B}}{L} \approx \mu_0 \tilde{j} \qquad \text{and} \qquad \frac{\tilde{E}}{L} \approx \frac{\tilde{\rho}}{\varepsilon_0}\,, \tag{5}$$

which leads to

$$\frac{c_L \tilde{B}}{\tilde{E}} \approx \frac{\tilde{j}}{\tilde{\rho} c_L} = \xi. \tag{6}$$

Hence, one has shown that assuming $E \gg c_L B$ ($E \ll c_L B$) is the consequence of assuming $\xi \ll 1$ ($\xi \gg 1$).

In addition, Ampère's equation leads to

$$\tilde{B} \approx \frac{\tilde{v}\tilde{E}}{c_L^2}\,, \tag{7}$$

where $\tilde{v} \approx L/\tau$ is the order of magnitude of a typical velocity of the system under consideration.

Faraday's equation gives

$$\tilde{E} \approx \tilde{v}\tilde{B}, \tag{8}$$

which are compatible only if $\tilde{v} \approx c_L$ (Lorentz-covariant electromagnetism).

As a consequence, either we have $\tilde{B} \approx \tilde{v}\tilde{E}/c_L^2$ which is compatible with $\nabla \times \boldsymbol{E} \approx 0$ (the time derivative of the magnetic field drops) and $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + 1/c_L^2 \partial_t \boldsymbol{E}$ (the electric limit) or we have $\tilde{E} \approx \tilde{v}\tilde{B}$ which is compatible with $\nabla \times \boldsymbol{B} \approx \mu_0 \boldsymbol{j}$ (the time derivative of the electric field drops) and $\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$ (the magnetic limit) [1].

Once again, we underline forcefully that we have only shown compatibility between some approximations of the full set of "Maxwell equations" with Galilean relativity as in [1]. We will now present what we think to be a demonstration of the two Galilean limits [1, 8, 10].

Indeed, the author has recently proposed to use the so-called Riemann-Lorenz formulation (the potentials are the basic quantities) instead of the so-called Heaviside-Hertz formulation (the fields are the basic quantities) in order to describe any experimental fact relative to Classical Electromagnetism [10]. The Riemann-Lorenz procedure consists in using the following postulate: "Any experimental fact of Classical Electromagnetism can be explained through the use of a scalar and a vector potential which are solutions of a set of Riemann equations with source terms (current density for the vector potential and charge density for the scalar potential) (9) assuming that both potentials are constrained to fulfill the Lorenz equation (10). With respect to the interaction with the matter, the Lorentz force usually written in terms of the fields can be rewritten in terms of the potentials (11)":

$$\nabla^2 V - \frac{1}{c_L^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla^2 \boldsymbol{A} - \frac{1}{c_L^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{j}\text{: Riemann equations,} \tag{9}$$

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c_L^2}\frac{\partial V}{\partial t} = 0\text{: Lorenz equation,} \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v} + q\boldsymbol{A}) = -\nabla(V - \boldsymbol{v}\cdot\boldsymbol{A})\text{: Lorentz force.} \tag{11}$$

The purpose of this article is not to discuss the validity of this postulate but to show what it implies with respect to Galilean electromagnetism using the potentials.

Assuming that the sources vanish at infinity, the potentials are expressed by the so-called retarded formulae:

$$V(M,t) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(P, t - PM/c_L)}{PM}\mathrm{d}\tau \quad \text{and} \quad \boldsymbol{A}(M,t) = \frac{\mu_0}{4\pi}\iiint \frac{\boldsymbol{j}(P, t - PM/c_L)}{PM}\mathrm{d}\tau. \tag{12}$$

We explicitly assume that the potentials are defined up to a constant which, for an infinite volume, is taken to be zero. If the volume of investigation is bounded like in a Faraday cage, the contribution of all the sources outside the volume resumes to a constant which is different from zero as can be shown easily with the Green formula.

In the quasi-static approximation, where $\varepsilon \ll 1$, the so-called retarded formulae (12) for the potentials become

$$V(M,t) \approx \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(P,t)}{PM} \mathrm{d}\tau \qquad \text{and} \qquad \boldsymbol{A}(M,t) \approx \frac{\mu_0}{4\pi} \iiint \frac{\boldsymbol{j}(P,t)}{PM} \mathrm{d}\tau. \tag{13}$$

These approximations are the solutions of Poisson equations for the potentials (14) which are the quasi-static limits of the Riemann equations with source terms [6]:

$$\nabla^2 V \approx -\frac{\rho}{\varepsilon_0} \qquad \text{and} \qquad \nabla^2 \boldsymbol{A} \approx -\mu_0 \boldsymbol{j}\,. \tag{14}$$

From this last remark, we can evaluate the order of magnitude of the potentials in function of the sources $\tilde{j}$ and $\tilde{\rho}$ which are given *a priori* (ϑ is the order of magnitude of the source volume):

$$\tilde{V} \approx \frac{1}{4\pi\varepsilon_0} \frac{\tilde{\rho}\vartheta}{L} \qquad \text{and} \qquad \tilde{A} \approx \frac{\mu_0}{4\pi} \frac{\tilde{j}\vartheta}{L}\,. \tag{15}$$

Contrary to Holland and Brown [3], we explicitly break gauge invariance of the Heaviside-Hertz formulation by giving orders of magnitude to the potentials. By saying that we can evaluate the order of magnitude of the potentials, we assume that we evaluate the order of magnitude of the potentials with respect to the constant on the boundary of the domain which is null if infinite and without sources at infinity. Hence, the tilde means the order of magnitude of a difference of potentials. Indeed, only the concept of difference of potential does have a physical meaning in the Riemann-Lorenz formulation. Yet, we point out forcefully that a difference of potential is not equal to a field: for example, the static field inside a capacitor is equal to the difference of potential between the two plates divided by the distance between them.

Now, one can form the following non-dimensional ratio using (15):

$$\frac{c_L \tilde{A}}{\tilde{V}} \approx \frac{c_L \mu_0 \tilde{j}}{\frac{\tilde{\rho}}{\varepsilon_0}} = \frac{\tilde{j}}{\tilde{\rho} c_L} = \xi. \tag{16}$$

We would like to know what the Lorenz gauge condition as well as the charge conservation $\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial \boldsymbol{t}} = 0$ become within the Galilean limits. We evaluate the orders of magnitude (double vertical lines) of each component of the spatial terms in these equations with respect to the temporal term:

$$\frac{\|\nabla \cdot \boldsymbol{A}\|}{\|\frac{1}{c_L^2}\frac{\partial V}{\partial t}\|} \approx \frac{\frac{\tilde{A}}{L}}{\frac{\tilde{V}}{c_L^2 \tau}} \approx \frac{c_L \tau}{L} \frac{c_L \tilde{A}}{\tilde{V}} = \frac{\xi}{\varepsilon} \qquad \text{and} \qquad \frac{\|\nabla \cdot \boldsymbol{j}\|}{\|\frac{\partial \rho}{\partial t}\|} \approx \frac{\frac{\tilde{j}}{L}}{\frac{\tilde{\rho}}{\tau}} \approx \frac{c_L \tau}{L} \frac{\tilde{j}}{c_L \tilde{\rho}} = \frac{\xi}{\varepsilon}\,. \tag{17}$$

As one can see, the same ratio between ε and ξ is implied. Now, according to Lévy-Leblond and Le Bellac, the quadri-current has the following Galilean limits [1]:

Electric limit:	Magnetic limit:
$\rho' = \rho$ and $\boldsymbol{j}' = \boldsymbol{j} - \rho\boldsymbol{v}$	$\rho' = \rho - \frac{\boldsymbol{v}\cdot\boldsymbol{j}}{c_L^2}$ and $\boldsymbol{j}' = \boldsymbol{j}$
which leads to $\xi_e = \frac{\tilde{j}}{\tilde{\rho} c_L} \approx \frac{\tilde{\rho}\tilde{v}}{\tilde{\rho} c_L} \approx \varepsilon$	which leads to $\xi_m = \frac{\tilde{j}}{\tilde{\rho} c_L} \approx \frac{\tilde{j}}{\frac{\tilde{v}\tilde{j}}{c_L^2} c_L} \approx \frac{1}{\varepsilon}$

Hence, ξ is different whether one considers the electric or the magnetic limit.

For Lorentz covariant electromagnetism, we have obviously $\varepsilon \approx \mathrm{O}(1)$ and $\xi \approx \mathrm{O}(1)$, which implies that the two terms in the Lorenz gauge are of the same order of magnitude: Lorenz gauge condition is Lorentz covariant which is well known.

In the quasi-static approximation, where $\varepsilon \ll 1$, we get

$$\text{Electric limit:} \qquad \xi_e \approx \varepsilon \ll 1 \qquad \text{and} \qquad \frac{\xi_e}{\varepsilon} \approx \mathrm{O}(1). \tag{18}$$

According to (18), the Lorenz gauge $\nabla \cdot \boldsymbol{A} + \frac{1}{c_L^2}\frac{\partial V}{\partial t} = 0$ is now Galilean covariant with respect to the electric transformations of the potentials,

$$\text{Magnetic limit:} \qquad \xi_m \approx \frac{1}{\varepsilon} \gg 1 \qquad \text{and} \qquad \frac{\xi_m}{\varepsilon} \gg 1. \tag{19}$$

Hence, using (19), the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ is the approximation of the Lorenz gauge within the magnetic limit and is now Galilean covariant with respect to the magnetic transformations of the potentials [8, 10]. The same conclusion applies for the charge conservation: $\nabla \cdot \boldsymbol{j} = 0$ is the Galilean magnetic limit of $\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0$ which is both Lorentz-covariant and covariant with respect to the Galilean electric limit [1, 8]. In addition, we point out forcefully that $\nabla \cdot \boldsymbol{j} = 0$ does not mean that the current density is static. Indeed, a time-dependent generator related to a lamp is such that the current density is divergenceless but varies in time: it is just the local expression of the global Kirchhof's law for time-dependent current intensities. This remark is well known in magnetohydrodynamics [4].

Using the Poisson equations for the potentials and either the Lorenz or the Coulomb gauge depending on the electric or the magnetic limit, one can easily derive the two sets of Galilean Maxwell equations for the fields proposed by Lévy-Leblond and Le Bellac [1]. The important point is to recognize that the two Galilean sets of equations in terms of the fields were stated without demonstration in [1] whereas here, we can demonstrate them starting with the potentials. We emphasized that Lévy-Leblond and Le Bellac procedure is completely valid, despite the fact that it relied on an assumption with respect to the relative importance of the electric field and the magnetic field, which remained to be justified as we have done in this paper using the potentials. In addition, we think we have resolved a long-standing problem in Classical Electromagnetism, that is we have provided physical motivations in order to choose either the Lorenz equation or the Coulomb equation depending on the Galilean or relativistic features (encoded in the scaling parameters) of the problem we are dealing with.

Conclusion. – As a conclusion, we have shown that the Lorenz equation applies in both Lorentz-covariant relativity and Galilean-covariant electric limit of Lévy-Leblond and Le Bellac whereas the Coulomb equation applies only within the Galilean-covariant magnetic limit [8, 10]. We have explicitly broken gauge invariance in order to get these results in accordance with the Riemann-Lorenz formulation of Classical Electromagnetism [10]. This last fact is *a priori* astonishing and contradictory, but it was demonstrated long ago that Galilean covariance and gauge invariance were incompatible [11]. Galilean electromagnetism is an unexpected field of actual research as we need to explore all its consequences in our current understanding of the special theory of relativity. As recalled recently by Norton, this theory emerged from Albert Einstein's struggle with the Maxwell-Lorentz's pre-1905 electromagnetic theory, which is a mixing of the magnetic and the electric limits without the essential property of group additivity and which made it untenable [12].

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