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Review

Forty years of Galilean Electromagnetism (1973–2013)

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Abstract. We review Galilean Electromagnetism since the 1973 seminal paper of Jean-Marc Lévy-Leblond and Michel Le Bellac and we explain for the first time all the historical experiments of Rowland, Vasilescu Karpen, Roentgen, Eichenwald, Wilson, Wilson and Wilson, which were previously interpreted in a Special Relativistic framework by showing the uselessness of the latter for setups involving slow motions of a part of the apparatus. Galilean Electromagnetism is not an alternative to Special Relavity but is precisely its low-velocity limit in Classical Electromagnetism.

Introduction

Niederle and Nikitin stated recently that [1] "analyzing contents of the main impact journals in theoretical and mathematical physics one finds that an interest of research in Galilean aspects of electrodynamics belongs to an evergreen subject". Despite the long history of the electrodynamics of moving media as exemplified by the world-famous paper by Albert Einstein entitled *On the electrodynamics of moving bodies* [2,3], several questions remain to be answered [4–25]. The Abraham-Minkowski controversy about the correct expression of the energy-momentum tensor in matter is a well-known example [26–29]. The interplay between moving fluids (normal, ferrofluids, liquid crystals...) and applied fields still generates interest (see, for instance, refs. [30–32]). In addition, the validity of the Lorentz transformations, when applied to rotation and non-uniform motion, is at the center of a vivid debate [33–42,32,43,44]. The goal of this work is to revisit some historical experiments which supported Special Relativity, from certainly a relativistic point of view but a Galilean one; we use "Galilean Electromagnetism", first considered in 1973 by Le Bellac and Lévy-Leblond (LBLL) [45] and re-examined in [8,46–59]. In this review paper, we underline first the recent achievements of Galilean Electromagnetism with a historical perspective. After, we recollect the relativistic description of Minkowski's electrodynamics without using tensor analysis. Then, we give a technical summary of some recent results on the Galilean electrodynamics of moving media. Afterwards, the Galilean constitutive equations are presented and, when necessary, they are used to explain the Rowland, Vasilescu Karpen, Roentgen, Eichenwald, Wilson, Wilson and Wilson's experiments without any recourse to Special Relativity. Our purpose is to show that coherently defined Galilean theories can describe experiments, otherwise understood as being "relativistic effects". Let us underline strongly that Galilean Electromagnetism is not an alternative to Special Relavity that remains unchanged but Galilean Electromagnetism is precisely the low-velocity limit of Special Relativity when applied to Classical Electromagnetism.

1 The revision of Classical Electromagnetism via its Galilean limits

According to Jean-Marc Lévy-Leblond, "the ideas, no more than the beings, are not born grown-up. It is rather in the confusion than they appear at first, embarrassed by the notions which they are going to invalidate, and formulated in inappropriate and soon void terms. That is why the "scientific revolutions" are not enough for the march of our knowledges; it is necessary that succeed them time of "revision" (Bachelard), who allow the purge, the (temporary) stabilization and the reformulation of the new theories".

It seems today that Classical Electromagnetism is in a phase of revision. Classical Electromagnetism is a theoretical corpus of experimental facts and interpretations stemming from the unification of the sciences of electricity and

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magnetism via the principle of relativity. It is to the Scot James Clerk Maxwell that we owe in 1861 the writing of a set of equations describing electromagnetic phenomena [60]. Henri Poincaré, reader of Clerk Maxwell, formulated in 1904 the principle of relativity [61]: the equations of physics, the mathematical translation of experimental facts, keep the same form whatever are the observers in uniform relative movements with regard to others. We speak about a principle of covariance to qualify its "democratic" character. Albert Einstein, reader of Poincaré, formulated in 1905 the following principle of invariance [2,3]: light, the mediator of information, has a constant velocity, independent of the source velocity. Thus, the light has a particular, "anti-democratic" status. Hermann Minkowski, reader of Poincaré and Einstein, suggested in 1908 to unify space and time in a continuum space-time [62]. Poincaré and Minkowski introduced the notion of the four-vector, a mathematical object varying as a space-time transformation (said "of Lorentz") in a change of inertial frame of reference, formed by a "spatial" vector part (for example, the position, the density of current, the vector potential) and of a scalar "temporal" part (for example, the time, the density of charge, the scalar potential). Four-vectors display, under a manifestly covariant form, the equations of physics. Electricity and magnetism are no more than appearances in a certain frame of reference of observation of the unifying entity called the electromagnetic field. Although the modern groups theory and the notion of causality allow to express mathematically the laws of transformations for space-time from one inertial frame of reference to another one, by using only the principle of covariance and by creating a constant of structure (mediator of the information, having the dimension of a speed), it seems that the principle of invariance clarifies indirectly how is made the taking of the Galilean limit because the light loses then its privileged status (its speed would differ between two frames of reference which is in contradiction, naturally, with the experimental facts).

Jean-Marc Lévy-Leblond showed in 1965 that the Lorentz transformations degenerate towards the transformations of Galilee provided one makes two hypotheses [63]: the relative speed between two inertial frames of reference is very small with regard to the speed of light (this last one remaining finite, to make it tend towards infinity has no sense because it would be counter-factual); a Galilean phenomenon takes place in an arena, the spatial extension of which is very small with regard to the distance covered by light during the duration of the phenomenon. If the second condition is relaxed, we can show that an a-causal limit (the so-called Carroll kinematics) at low speeds is completely possible mathematically but Lévy-Leblond excluded it by the physical requirement of causality [63]. The important point of the taking of a Galilean limit is that the spatial part of the four-vector "position" in four dimensions is smaller than its temporal part. Einstein's mechanics, which describes the motion of massive particles, admits one single Galilean limit, that is Newton's mechanics. What about Classical Electromagnetism?

The modern presentation of Special Relativity often consists in the following fable: "At the end of the 19th and at the beginning of 20th centuries, some famous physicists noticed the incompatibility between, on the one hand, the mechanics of Newton and, on the other hand, the electromagnetism of Maxwell. In particular, the equations of Maxwell (where c represents an invariant, the speed of light) are not covariant according to the transformations of Galileo of space and time. A new mechanics was so created by generalizing that of Newton to be compatible with both principles of covariance (known since Galileo for the mechanics and generalized by Poincaré for the whole physics) and of invariance (dictated by the equations of Maxwell and, for example, the experiment of interferences optics of moving bodies due to Michelson and Morley)."

The success of Special Relativity followed by General Relativity was unprecedented and we can qualify *a posteriori* this theory of a scientific revolution. However, a dogma appeared: "Classical Electromagnetism is incompatible with Galilean physics". It was necessary to wait until 1973, when Jean-Marc Lévy-Leblond, supported by Michel Le Bellac, asked the following relatively naive but brave question [45]: if Einsteinian mechanics has a Galilean limit, why does not Electromagnetism? We saw that taking the limit features two stages: limitation to a regime of low speeds and comparison of the spatial and temporal parts of the envisaged four-vector. In Classical Electromagnetism, there is no reason for postulating that the spatial part is always smaller than the temporal part. It is true for the four-vector position but groundless generally. For example, the spatial part of the four-current, *i.e.* the density of electric current can be much bigger than its temporal part, *i.e.* the density of electric charge: it is what takes place, for example, in an ohmic conductor gone through by a current where the charge density is null. So, Classical Electromagnetism admits two low-velocity limits! A revision is thus necessary.

In 1973, Lévy-Leblond and Le Bellac postulated two sets of approximate Maxwell equations compatible with the Galilean transformations of both sources and fields and deducted from their taking of limit [45]. They distinguished the "magnetic" said Galilean limit, which applies to magnets where the so-called displacement current is neglected in Maxwell's equations, and the "electric" said limit, which applies to insulators where the term of Faraday induction disappears. In 2003, Marc de Montigny [50] and the present author [51] demonstrated independently both sets of approximate Maxwell's equations postulated by Lévy-Leblond and Bellac. Marc de Montigny used groups theory with a tensorial Lagrangian formulation in five dimensions (the fifth constituent being the action) followed by a process said of "reduction" [50]. The present author used a reasoning with orders of magnitude to write the Galilean limits of the Lorentz transformations of the electromagnetic potentials and of the so-called "gauge conditions" [51]. In particular, the Canadian and French researchers showed that the Lorenz "gauge condition" is at once compatible with the Lorentz transformations and the electric Galilean transformations while the Coulomb "gauge condition" applies only in the case of the magnetic limit [50–54]. So, the mystery of the ranges of validity of the "gauge conditions" was solved by the recognition of their relativist or Galilean character depending on the context. The present author had been led towards this result by noticing the analogy between the "gauge conditions" and the mass continuity equation for a fluid in motion. The analogy between Fluid Mechanics and Classical Electromagnetism is the one introduced by James Clerk Maxwell to derive his famous set of equations [60]. The Coulomb "gauge condition" (similar to the constraint of incompressibility) is the low-speed limit of the Lorenz "gauge condition" (similar to the constraint of compressibility for the acoustic waves) [51]. In 2013, Giovanni Manfredi derived the same conclusions with respect to the range of validity of the gauge condition by providing a systematic derivation of these two limits based on a dimensionless form of Maxwell's equations and an expansion of the electric and magnetic fields in a power series of some small parameters [58]. He extended a procedure introduced by Melcher but that was applied to fields only [64,24] (see also [59] for applications to condensers and solenoids).

The reader may have been worried by the fact that the expression "gauge condition" is written with quotation marks. Indeed, they are not mathematical equations taken without physical motivation but they are true physical constraints, namely continuity equations with mechanical analogues [51,65]. It has been explained elsewhere why the four-potential is a physical quantity contrary to the old-established belief which dismisses a physical interpretation to the potentials of Classical Electromagnetism [51,54,65]. It has been also explained why the "gauge conditions" are physical constraints contrary to the same old-established belief [51,65]. The vector potential was measured very recently in a classical context with a quantum probe [66]. Its necessity to explain a classical experiment was shown [65] (see another example in [67]). Nobel prize winners have discussed recently the reality of the vector potential [68–70].

2 Minkowski electrodynamics, Poincaré covariance and constitutive equations

After Poincaré and Einstein have proved, in 1905, the covariance of the full set of Maxwell's equations, including the case of sources in vacuum [2,3], the extension to continuous media, including polarization and magnetization effects, was masterly tackled by Hermann Minkowski in 1908 [62]. As a leading mathematician of his time, Minkowski formulated special relativity with the tools of tensorial analysis. Indeed, he followed the path outlined by Poincaré (who introduced the four-vectors) by introducing what he called "vectors of the first species" (*i.e.* four-vectors, like the charge and current densities), as well as "vectors of the second species" (*i.e.* hexa-vectors like the one formed by the electric and induction fields). His terminology is no longer used today but the transformations properties of the associated tensors have become commonplace. Minkowski was the first to realize that the constitutive equations were not covariant under a Lorentz transformation [62]; one supposes their validity in the moving frame and then expresses them in the laboratory frame. We will outline this process by adopting the latter presentation of Einstein and Laub introduced in 1908 [71–75], without recourse to tensors, as discussed by Pauli in his review article of 1921, cited in [12].

The relativistic form of Maxwell's equations (also referred to as "Maxwell-Minkowski equations") in continuous media is written as

$$\begin{aligned} \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, && \text{Faraday,} \\ \nabla \cdot \mathbf{B} &= 0, && \text{Thomson,} \\ \nabla \times \mathbf{H} &= \mathbf{j} + \partial_t \mathbf{D}, && \text{Ampère,} \\ \nabla \cdot \mathbf{D} &= \rho, && \text{Gauss.} \end{aligned} \tag{1}$$

A Lorentz transformation acts on space-time coordinates as follows (see, for instance, sect. 7.2 of [76]):

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \gamma \mathbf{v} t + (\gamma - 1)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{x})}{\mathbf{v}^2}, \\ t' &= \gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right), \end{aligned} \tag{2}$$

where $\mathbf{v}$ is the relative velocity and $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$. Under this transformation, the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$, and their respective inductions, $\mathbf{D}$ and $\mathbf{B}$, transform as follows:

$$\begin{aligned} \mathbf{E}' &= \gamma\left(\mathbf{E} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{\mathbf{v}^2} + \mathbf{v}\times\mathbf{B}\right), \\ \mathbf{B}' &= \gamma\left(\mathbf{B} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{B})}{\mathbf{v}^2} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right), \\ \mathbf{D}' &= \gamma\left(\mathbf{D} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{D})}{\mathbf{v}^2} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H}\right), \\ \mathbf{H}' &= \gamma\left(\mathbf{H} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{H})}{\mathbf{v}^2} - \mathbf{v}\times\mathbf{D}\right). \end{aligned} \tag{3}$$

We consider a medium in motion that is linear, homogeneous and isotropic. We denote its permittivity by ϵ and its permeability by μ. Hence, the constitutive equations in the moving frame (Minkowski's crucial hypothesis [62]),

$$\begin{aligned} \mathbf{D}' &= \epsilon \mathbf{E}', \\ \mathbf{B}' &= \mu \mathbf{H}', \end{aligned} \tag{4}$$

become

$$\begin{aligned} \mathbf{D} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{D})}{\mathbf{v}^2} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H} &= \epsilon\left(\mathbf{E} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{\mathbf{v}^2} + \mathbf{v}\times\mathbf{B}\right), \\ \mathbf{B} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{B})}{\mathbf{v}^2} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E} &= \mu\left(\mathbf{H} - \frac{(\gamma-1)}{\gamma}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{H})}{\mathbf{v}^2} - \mathbf{v}\times\mathbf{D}\right). \end{aligned} \tag{5}$$

The scalar product with the velocity $\mathbf{v}$ of eq. (5) gives

$$\begin{aligned} \mathbf{D}\cdot\mathbf{v} &= \epsilon\mathbf{E}\cdot\mathbf{v}, \\ \mathbf{B}\cdot\mathbf{v} &= \mu\mathbf{H}\cdot\mathbf{v}, \end{aligned} \tag{6}$$

which allows us to simplify eq. (5) to the following expression:

$$\begin{aligned} \mathbf{D} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H} &= \epsilon(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \\ \mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E} &= \mu(\mathbf{H} - \mathbf{v}\times\mathbf{D}). \end{aligned} \tag{7}$$

The latter relativistic constitutive equations were first written in 1908 by Minkowski in his groundbreaking paper [62].

If we write $\mathbf{D}$ and $\mathbf{H}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ by using eq. (6), and utilize the formula for the double vectorial product,

$$\begin{aligned} \mathbf{v}\times(\mathbf{v}\times\mathbf{D}) &= \mathbf{v}(\mathbf{v}\cdot\mathbf{D}) - \mathbf{v}^2\mathbf{D}, \\ \mathbf{v}\times(\mathbf{v}\times\mathbf{E}) &= \mathbf{v}(\mathbf{v}\cdot\mathbf{E}) - \mathbf{v}^2\mathbf{E}, \end{aligned} \tag{8}$$

we obtain the following relativistic expressions in the laboratory rest frame:

$$\begin{aligned} \mathbf{D} &= \gamma^2\epsilon\left[\left(1 - \frac{\mathbf{v}^2}{\mu\epsilon c^4}\right)\mathbf{E} + \left(1 - \frac{1}{\mu\epsilon c^2}\right)\left(\mathbf{v}\times\mathbf{B} - \frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}\right)\right)\right], \\ \mathbf{H} &= \frac{\gamma^2}{\mu}\left[(1-\mu\epsilon\mathbf{v}^2)\mathbf{B} + \left(\mu\epsilon - \frac{1}{c^2}\right)(\mathbf{v}\times\mathbf{E} + \mathbf{v}(\mathbf{v}\cdot\mathbf{B}))\right]. \end{aligned} \tag{9}$$

Other useful formulae, derived from eqs. (7), are

$$\begin{aligned} \mathbf{B} &= \frac{1}{1-\mu\epsilon v^2}\left[\mu\left(1-\frac{v^2}{c^2}\right)\mathbf{H} - \left(\mu\epsilon - \frac{1}{c^2}\right)\mathbf{v}\times\mathbf{E} + \mu\left(\mu\epsilon - \frac{1}{c^2}\right)(\mathbf{v}\cdot\mathbf{H})\mathbf{v}\right], \\ \mathbf{D} &= \frac{1}{1-\mu\epsilon v^2}\left[\epsilon\left(1-\frac{v^2}{c^2}\right)\mathbf{E} + \left(\mu\epsilon - \frac{1}{c^2}\right)\mathbf{v}\times\mathbf{H} - \epsilon\left(\mu\epsilon - \frac{1}{c^2}\right)(\mathbf{v}\cdot\mathbf{E})\mathbf{v}\right]. \end{aligned} \tag{10}$$

Finally, eqs. (6), (7) and (8) lead to

$$\begin{aligned} \mathbf{D} &= \epsilon\mathbf{E} + \gamma^2\left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times\left(\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right), \\ \mathbf{H} &= \frac{\mathbf{B}}{\mu} + \gamma^2\left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \end{aligned} \tag{11}$$

These are the forms of the constitutive equations more amenable with a Galilean limit. One recovers the usual constitutive relations either for vacuum with motion ($\mu = \mu_0$, $\epsilon = \epsilon_0$ and $\mathbf{v} \neq 0$) or for media at rest ($\mu \neq \mu_0$, $\epsilon \neq \epsilon_0$ and $\mathbf{v} = 0$).

3 Electrodynamics of continuous media at low velocities

According to Le Bellac and Lévy-Leblond [45,49,51–55], any four-vector $(u_0, \mathbf{u})$ has two Galilean low-velocity limits depending on the relative magnitude between its spatial and temporal parts. For example, let v denotes a typical velocity of the system under study (it can be a true velocity: say the one of a moving magnet with respect to the laboratory frame or a fictive one like the product of the radius of a solenoid and the working frequency of the varying current flowing in it). A Galilean limit is such that $v \ll c$ where c is the light velocity [45,77–82]. A time-like (space-like) Galilean limit implies, in addition, that $u_0 \gg u$ ($u_0 \ll u$). From the Lorentz transformation of the four-vector, we can derive two limits which are the time-like Galilean transformations $[u_0' = u_0, \mathbf{u}' = \mathbf{u} - \frac{u_0}{c}\mathbf{v}]$ and the space-like Galilean transformations $[u_0' = u_0 - \frac{1}{c}\mathbf{v}\cdot\mathbf{u}, \mathbf{u}' = \mathbf{u}]$. For any four-vector, one deduces two low-velocity approximations from the "relativistic" transformation where the Lorentz contraction factor has disappeared but where we kept the constraint of group additivity which encodes the (Special) Principle of Relativity. As recalled by Heras, the status of c changes when one takes a Galilean limit [56,57]. Indeed, since Maxwell [60], c is usually both a unit "translator" c_u (obtained from comparing the unit of force for the quasi-static laws of Coulomb and Biot and Savart) and the velocity of light c_L. When, in a Galilean transformation, one writes c, it means c_u and not c_L since c_L is irrelevant in the low-velocity approximation $v \ll c_L$ [45,77–82]. For simplicity, we will keep c instead of c_u in the Galilean transformations.

Here, let us underline the difference between covariance and invariance [83]: c_L is a Lorentz invariant; u_0 is a Galilean invariant in the time-like limit but not in the space-like limit. The equations of "motion" (be it Maxwell's equations or the equations for the potentials) must keep the same form when changing from one inertial frame to another: they are covariant with respect to the space-time transformations (be it Lorentz or Galilean transformations). Covariance is the mathematical expression of the Physical (Special) Principe of Relativity.

Now, we have to find four-vectors in order to apply the Galilean reduction (the couple $(\mathbf{E}, c\mathbf{B})$ is not):

- Is the couple $(\rho c, \mathbf{J})$ a four-vector? According to Poincaré [61], yes if and only if the couple is constrained by the charge conservation $\nabla\cdot\mathbf{J} + \partial_t\rho = 0$. Indeed, the charge continuity equation is a four-scalar product $\partial_\mu J^\mu = 0$, that is, it is an invariant provided the couple $(\rho c, \mathbf{J})$ does constitute a four-vector.
- Is the couple $(V, c\mathbf{A})$ a four-vector? According to Poincaré [61], yes if and only if the couple is constrained by the Lorenz "gauge condition" $\nabla\cdot(c\mathbf{A}) + \partial_t(V/c) = 0$. Indeed, the Lorenz continuity equation is a four-scalar product $\partial_\mu A^\mu = 0$, that is, it is an invariant provided the couple $(V, c\mathbf{A})$ does constitute a four-vector. The Coulomb "gauge condition" $\nabla\cdot\mathbf{A} = 0$ is *not* Lorentz-covariant.

What are the Galilean limits of the Lorentz transformations of the four-potential? The time-like limit ($V \gg cA$) transformations are $V' = V$ and $\mathbf{A}' = \mathbf{A} - V/c^2\mathbf{v}$ whereas the space-like limit ($V \ll cA$) transformations are $V' = V - \mathbf{v}\cdot\mathbf{A}$ and $\mathbf{A}' = \mathbf{A}$ with $\mathbf{v}$ the relative velocity. The former transformations are known as the electric limit whereas the latter constitute the magnetic limit. Hence, from these limits one easily deduces (we use $\partial_t' = \partial_t - \mathbf{v}\cdot\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}' = \boldsymbol{\nabla}$) [51–54] what follows.

- The Lorenz "gauge condition" is covariant with respect to the Galilean electric limit whereas it is not for the magnetic limit.
- The Coulomb "gauge condition" is covariant with respect to the Galilean magnetic limit whereas it is not for the electric limit.
- The Coulomb "gauge condition" is the Galilean magnetic limit of the Lorentz-covariant Lorenz "gauge condition". These two "gauge conditions" are not independent. One is the quasi-stationnary (and obviously the stationary) limit of the other.
- The Lorenz "gauge condition" has a double status since it is both compatible with the Lorentz transformations and the Galilean electric transformations

Let us recall that the same conclusions were obtained independently by de Montigny *et al.* in a very different way using group theory [50] and by Manfredi using power series expansion [58]. To have several demonstrations of the same result is an indication of its robustness...

One does not need to assume simultaneously $E \gg cB$ and $\rho c \gg J$ in order to derive, for example, the electric limit. $E \gg cB$ is a consequence of $\rho c \gg J$. Similarly, $V \gg cA$ is a consequence of $\rho c \gg J$. As a matter of fact, both potentials are solutions of a Poisson equation in the Galilean limits: hence, the ratio $J/(\rho c)$ has the same limits as the ratio $(cA)/V$ [54]. The electric limit transformations for the potentials $V' = V$ and $\mathbf{A}' = \mathbf{A} - V/c^2\mathbf{v}$ are incompatible with the Coulomb "gauge condition" since the latter is not covariant with respect to these transformations. When one wants to use an equation in a specific context, one has to check if this equation is compatible with the underlying space-time symmetry. Moreover, the electric limit implies $cA \ll V$ in addition to the low-velocity approximation $v \ll c$: these two constraints lead necessarily to the fact that both terms in the Lorenz "gauge condition" are of the same order of magnitude [54]. Hence, one cannot drop the temporal term with respect to the divergence term in the Lorenz "gauge condition" to get the Coulomb "gauge condition" within the electric limit (which would be the case in the magnetic limit because $cA \gg V$). It is obvious that within the magnetic limit, some solutions are such that the scalar potential vanishes. One can think of a solenoid in the laboratory frame, the scalar potential is zero since there is no charge density in the laboratory frame. This will not remain true in a moving frame. To impose that the scalar potential vanishes is a consequence of the cancellation of the charge density. Physical reasoning never forces the scalar potential to be zero without having introduced a hypothesis on its source. Mathematical reasoning resorts to tricks like "gauge transformations" to impose such an unphysical statement.

In ref. [53], we started with the two postulated and approximate Galilean sets of Maxwell-Minkowski equations in continuous media, and we obtained the following field transformations for a continuous medium moving with a small velocity compared to light (Galilean approximation):

$$\begin{array}{cc}
\text{Magnetic limit} & \text{Electric limit} \\
\rho_m = \rho'_m + \mathbf{v}\cdot\mathbf{j}'_m/c^2, & \rho_e = \rho'_e, \\
\mathbf{j}_m = \mathbf{j}'_m, & \mathbf{j}_e = \mathbf{j}'_e - \rho'_e\mathbf{v}, \\
\mathbf{B}_m = \mathbf{B}'_m, & \mathbf{B} = \mathbf{B}'_e + \mathbf{v}\times\mathbf{E}'_e/c^2, \\
\mathbf{E}_m = \mathbf{E}'_m - \mathbf{v}\times\mathbf{B}'_m, & \mathbf{E}_e = \mathbf{E}'_e, \\
\mathbf{H}_m = \mathbf{H}'_m, & \mathbf{H}_e = \mathbf{H}'_e + \mathbf{v}\times\mathbf{D}'_e, \\
\mathbf{D}_m = \mathbf{D}'_m - \mathbf{v}\times\mathbf{H}'_m/c^2, & \mathbf{D}_e = \mathbf{D}'_e, \\
\mathbf{M}_m = \mathbf{M}'_m, & \mathbf{M}_e = \mathbf{M}'_e - \mathbf{v}\times\mathbf{P}'_e, \\
\mathbf{P}_m = \mathbf{P}'_m + \mathbf{v}\times\mathbf{M}'_m/c^2, & \mathbf{P}_e = \mathbf{P}'_e.
\end{array}$$

Most of these transformations are well known to electrical engineers. According to us, the oldest reference to them is the book by Woodson and Melcher in 1968, cited in [21]. However, the associated sets of approximate Maxwell equations were postulated up to now and the Galilean constitutive equations have not been written so far. For example, the two approximate sets have been used separately since a long time in electrohydrodynamics for the electric limit and in magnetohydrodynamics for the magnetic limit.

The engineering approach differs from the physicists' approach first introduced by Le Bellac and Levy-Leblond in 1973 [45] since the last authors focused on the Galilean limits of four-vectors starting from the Lorentz transformations whereas the engineers applied directly the galilean transformations of space-time to approximate sets of equations relying on orders of magnitude and physical consideration about the relative magnitude between the magnetic diffusion time, the charge relaxation time and the electromagnetic waves transit time [24].

In ref. [54], we demonstrated both sets of approximation starting from the relativistic theory and using the potentials formulation of Electromagnetism by pointing out the crucial role of the "gauge conditions". In addition, we recalled the boundary conditions for moving media (see [84–88] as well), with $\mathbf{n}$ being the unit vector between two media denoted by the superscripts 1 and 2, $\mathbf{K}$ the density of surface current sheet, σ the surface charge density, Σ the surface separating both media, and v_n the projection of the relative velocity on the normal of Σ,

$$\begin{array}{cc}
\text{Magnetic limit} & \text{Electric limit} \\
\mathbf{n}\times(\mathbf{H}^2_m - \mathbf{H}^1_m) = \mathbf{K}, & \mathbf{n}\times(\mathbf{E}^2_e - \mathbf{E}^1_e) = 0, \\
\mathbf{n}\cdot(\mathbf{B}^2_m - \mathbf{B}^1_m) = \mathbf{0}, & \mathbf{n}\cdot(\mathbf{D}^2_e - \mathbf{D}^1_e) = \sigma, \\
\mathbf{n}\cdot(\mathbf{j}^2_m - \mathbf{j}^1_m) + \nabla_\Sigma\cdot\mathbf{K} = \mathbf{0}, & \mathbf{n}\cdot(\mathbf{j}^2_e - \mathbf{j}^1_e) + \nabla_\Sigma\cdot\mathbf{K} = v_n(\rho^2_e - \rho^1_e) - \partial_t\sigma, \\
\mathbf{n}\times(\mathbf{E}^2_m - \mathbf{E}^1_m) = v_n(\mathbf{B}^2_m - \mathbf{B}^1_m), & \mathbf{n}\times(\mathbf{H}^2_e - \mathbf{H}^1_e) = \mathbf{K} + v_n\mathbf{n}\times[\mathbf{n}\times(\mathbf{D}^2_e - \mathbf{D}^1_e)].
\end{array}$$

4 Galilean constitutive equations

Now, starting with the postulate set of "fully relativistic" Maxwell-Minkowski equations [62], we will use orders of magnitude in order to derive the two approximate Galilean constitutive equations for both excitation fields.

As a very large part of the physics community is unaware of the existence of Galilean Electromagnetism, the Galilean constitutive relations will be derived from the known Relativistic Maxwell-Minkowski theory. Then, it will be straightforward to derive them from the Galilean transformations of the fields/inductions (see the previous section) thanks to the two postulated Galilean sets of Maxwell's equations. So, we can avoid, in the end, the Lorentz group completely.

Following the procedure adopted in refs. [52–55], a Galilean limit is obtained in two steps. First, we introduce into eqs. (11) the quasi-static approximation $v \ll c$ [77–82]. This assumption leads to equations which do not obey the group additivity property (and are clearly not Galilean covariant),

$$\mathbf{D} \simeq \epsilon\mathbf{E} + \left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times(\mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}),$$
$$\mathbf{H} \simeq \frac{\mathbf{B}}{\mu} + \left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{12}$$

At this stage, only the Fitzgerald-Lorentz contraction factor is set equal to unity. Next, an assumption on the relative magnitude of the remaining terms is made in order to drop the terms which break the Galilean covariance. One can see that the magnetic limit corresponds to the assumption $E_m \sim vB_m \ll cB_m$ (see refs. [52–55]). Hence, the Galilean magnetic constitutive equations are

$$\begin{aligned} \mathbf{D}_m &\simeq \epsilon \mathbf{E}_m + \left(\epsilon - \frac{1}{\mu c^2}\right) \mathbf{v} \times \mathbf{B}_m, \\ \mathbf{H}_m &\simeq \frac{\mathbf{B}_m}{\mu}. \end{aligned} \tag{13}$$

The electric limit, which corresponds to $cB_e \sim vE_e/c \ll E_e$ (see refs. [52–55]), leads to the following Galilean electric constitutive equations:

$$\begin{aligned} \mathbf{D}_e &\simeq \epsilon \mathbf{E}_e, \\ \mathbf{H}_e &\simeq \frac{\mathbf{B}_e}{\mu} + \left(\epsilon - \frac{1}{\mu c^2}\right) \frac{\mathbf{v} \times \mathbf{E}_e}{c^2}. \end{aligned} \tag{14}$$

A similar simplification could have been done with eq. (9) instead of eq. (11). Both sets (13) and (14) do form a transformation group. Moreover, they can be obtained directly from the Minkowski constitutive equations (7), since the magnetic limit transformations follow from

$$\begin{aligned} \mathbf{D}_m + \frac{1}{c^2}\mathbf{v} \times \mathbf{H}_m &\simeq \epsilon \left(\mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m\right), \\ \mathbf{B}_m &\simeq \mu \mathbf{H}_m, \end{aligned} \tag{15}$$

in accordance with the Galilean magnetic Maxwell-Minkowski equations,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E}_m &= -\partial_t \mathbf{B}_m, && \text{Faraday}, \\ \boldsymbol{\nabla} \cdot \mathbf{B}_m &= 0, && \text{Thomson}, \\ \boldsymbol{\nabla} \times \mathbf{H}_m &= \mathbf{j}_m, && \text{Ampère}, \\ \boldsymbol{\nabla} \cdot \mathbf{D}_m &= \rho_m, && \text{Gauss}. \end{aligned} \tag{16}$$

The electric limit transformations come from

$$\begin{aligned} \mathbf{D}_e &\simeq \epsilon \mathbf{E}_e, \\ \mathbf{B}_e - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}_e &\simeq \mu(\mathbf{H}_e - \mathbf{v} \times \mathbf{D}_e), \end{aligned} \tag{17}$$

in accordance with the Galilean electric Maxwell-Minkowski equations,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E}_e &= 0, && \text{Faraday}, \\ \boldsymbol{\nabla} \cdot \mathbf{B}_e &= 0, && \text{Thomson}, \\ \boldsymbol{\nabla} \times \mathbf{H}_e &= \mathbf{j}_e + \partial_t \mathbf{D}_e, && \text{Ampère}, \\ \boldsymbol{\nabla} \cdot \mathbf{D}_e &= \rho_e, && \text{Gauss}. \end{aligned} \tag{18}$$

It is now obvious to demonstrate the Galilean constitutive relations starting from the fields/inductions Galilean transformations recalled in sect. 2. Let us do it for the magnetic limit. We combine first both Galilean transformations $\mathbf{H}'_m = \mathbf{H}_m$ and $\mathbf{B}'_m = \mathbf{B}_m$ into the Minkowski's constitutive relation $\mathbf{B}'_m = \mu \mathbf{H}'_m$ in the moving frame to get the one in the static frame $\mathbf{B}_m = \mu \mathbf{H}_m$. Then, similarly using both $\mathbf{D}'_m = \mathbf{D}_m + \mathbf{v} \times \mathbf{H}_m/c^2$ and $\mathbf{E}'_m = \mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m$ into $\mathbf{D}'_m = \epsilon \mathbf{E}'_m$, one ends up with $\mathbf{D}_m \simeq \epsilon \mathbf{E}_m + (\epsilon - \frac{1}{\mu c^2})\mathbf{v} \times \mathbf{B}_m$ as expected. We point out forcefully that the Galilean constitutive relations can be derived either directly without using the Lorentz symmetry *or* by taking the quasi-static limit(s) of the Special Relativity theory.

In their seminal paper published in 1973, Le Bellac and Lévy-Leblond underlined that combinations of the electric and magnetic limits are of course possible [45]. Hence, the following Galilean displacement field is allowed:

$$\mathbf{D}_G = \mathbf{D}_m + \mathbf{D}_e \simeq \epsilon(\mathbf{E}_e + \mathbf{E}_m) + \left(\epsilon - \frac{1}{\mu c^2}\right) \mathbf{v} \times \mathbf{B}_m, \tag{19}$$

as we will see in a particular case ($\mathbf{E}_m = \mathbf{0}$) for the Wilson and Wilson's effect.

5 Rowland-Vasilescu Karpen's effect

In 1876, H.A. Rowland identified an equivalence between the conduction current and the convection current [89–95]. Indeed, he proved that the motion of electric charges has the same magnetic effect as a current given by Ohm's law within conductors. Rowland's effect is an example of the Galilean electric limit. The charges in motion satisfy the Galilean transformations $\mathbf{j}' = \mathbf{j} + \rho\mathbf{v}$, and $\rho' = \rho$.

A modern reproduction of this experiment consists in connecting a disk of hard rubber or an old phonograph record to the shaft of an electric motor. The disk is electrostatically charged by rubbing it with a piece of woolen cloth. Then, it is set in rotation and a magnetic compass is approached close to it. The needle is deflected; the faster the rotation, the greater the deflection.

The disk has a radius R and a thickness h. We assume that its volume is charged uniformly with a total charge Q. Hence, the volume charge density is $\rho = Q/(\pi R^2 h)$. Let us call $\mathrm{d}\tau$ the volume element of the disk between the radii r and $r + \mathrm{d}r$. When the disk is rotating at constant angular frequency ω, the volume $\mathrm{d}\tau$ carries out a charge $\rho\mathrm{d}\tau$ at a velocity $\mathbf{v} = r\omega\mathbf{e}_\theta$. Hence, the volume current density in the lab frame is $\mathbf{j} = -\rho\mathbf{v} = -Q/(\pi R^2 h)\mathbf{v}$ since there is no current density, that is, $\mathbf{j}' = \mathbf{0}$ in the frame of the disk. The equivalent current intensity $\mathrm{d}I$, which circulates in $\mathrm{d}\tau$ through the surface $\mathrm{d}S = h\mathrm{d}r$, is given by $\mathrm{d}I = \mathbf{j}\cdot\mathrm{d}\mathbf{S} = -Q/(\pi R^2 h)\omega r h\mathrm{d}r = -Q\omega r/(\pi R^2)\mathrm{d}r$. If one denotes the surface charge density by $\sigma = Q/(\pi R^2)$, then the equivalent current becomes $\mathrm{d}I = -\sigma\omega r\mathrm{d}r$. For a constant rotation, we can define the period $T = 2\pi/\omega$, and the equivalent current is given by the formula $\mathrm{d}I = -\sigma 2\pi r\mathrm{d}r/T$. Here, we see that is it not necessary to assume a distribution of charge in volume since the result depends on the surface distribution σ.

The disk is equivalent to a set of concentric rings each one carrying a current $\mathrm{d}I$. As is well known, a circular ring of current creates a magnetic induction at a perpendicular distance z from its center given by

$$\mathrm{d}\mathbf{B} = \frac{\mu_0\mathrm{d}I\sin^3(\phi)}{2r}\mathbf{e}_z, \tag{20}$$

where the angle ϕ is such that $\tan(\phi) = r/z$.

The integration over the entire disk is straightforward,

$$\mathbf{B} = -\frac{\mu_0\sigma\omega}{2}z\left(\frac{\sqrt{R^2+z^2}}{z} + \frac{z}{\sqrt{R^2+z^2}} - 2\right)\mathbf{e}_z. \tag{21}$$

When $R \ll z$, one can expand the preceding formula up to the fourth order in the small parameter R/z and obtain

$$\mathbf{B} \simeq -\frac{\mu_0\sigma\omega}{8}\frac{R^4}{z^3}\mathbf{e}_z. \tag{22}$$

Now, let us recall the expression for the magnetic induction produced by a magnetic dipole with moment $\mathbf{m} = m\mathbf{e}_z$,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{2m}{z^3}\mathbf{e}_z. \tag{23}$$

We conclude that a spinning charged disk is equivalent to a magnetic dipole of moment $m = -\pi\sigma\omega R^4/4$. One easily checks that $m = \int_{\mathrm{disk}}\mathrm{d}m = \int_{\mathrm{disk}}\pi r^2\mathrm{d}I$.

Originally, Rowland used a dielectric disk (ebonite) with a thin gold leaf on each side first [89]. Then, Rowland and Hutchinson utilized a metallic coating separated in sectors in order to avoid conduction currents [90]. Himstedt used glass with a surface treatment of lead. In both cases, a deflection of a metallic needle was observed [91].

In 1904, Nicolae Vasilescu Karpen defended his doctor's degree thesis by which he proved experimentally, with a high precision, that the magnetic field produced by the convection current is the same with the effect produced by the conduction current, in any conditions [96–98]. The doctor's degree examination commission (panel) was composed of Gabriel Lippmann, chairman, Henri Poincaré and Henri Moissan, members. For proving it, N. Vasilescu Karpen chose an indirect method based on the use of the electromotive force induced in a coil by electromagnetic induction. For this purpose, he conceived and achieved an apparatus having a rotating metalled disk (made of ebonite and covered with tinfoil on both parts) placed between two fixed metalled armatures (each being a rectangular glass blade covered also with tinfoil and having a central circular hole). The disk was connected through a system of moving contacts to one of the two terminals of an alternating current supply, whereas the two armatures were together connected to the other terminal of the alternating current supply. The disk and armatures were in vertical position. On each part of the disc, outside the armatures, a coaxial coil was mounted. The working circuit includes the coils and a capacitor. The disk was driven at the rotational speed of 200–800 rev/min by a direct-current motor. The electric charge on the rotating disk produces the convection current, therefore a magnetic field having a frequency like the supply voltage. Hence the electromotive force induced in the coils varies with the same frequency. By tuning the parameters of the working circuit to the frequency of the voltage supplying the apparatus, he obtained the greatest possible value of the current in the circuit, which could be measured with high precision, the results removing the doubts, which existed at that time [98].

6 Roentgen-Eichenwald's effect

The experiments of W.C. Roentgen (1885) and A. Eichenwald (1903) demonstrate that a dielectric which moves at a constant speed in an electric field produced a magnetic field due to the convection current of the moving induced surface charges [99–107]. All the fields are supposed to be stationary.

In the experiments of Roentgen, a disk made of a dielectric material rotates between two ring electrodes (each having the form of a plate with a hole at its center) at rest: the lower one is grounded whereas the upper one consists of two parts with opposite electric potentials. The sign of the polarization inside the dielectric changes two times per turn. Roentgen observed qualitatively the deflection of a magnetic needle due to the magnetic field created by the varying displacement field [99–104]. In 1888, he reported: "I rotated a round glass plane between two horizontal condenser plates (or a hard rubber plane), the upper of which was continuously derived to earth, the one below could be loaded with positive or negative electricity from a source of electricity. Close to the upper condenser plane hung one of two magnetic needles which were connected to a very sensitive system; their direction was vertical to a radius of the plane and its centre was above the plane next to the edge of it. The deviations of the needles, which occurred during the commutation of the condenser's load, could be observed with the help of a binocular, a mirror and a scale. These experiments showed that the needle was deviated every time during commutation; it was directed in such a way as if the direction of an allegedly existing current had been reversed. The effect of the movement of the positive poles to the needles corresponded with the flow of a current, flowing in the same direction, the movement of the negative poles that of a current, flowing in the opposite direction". It was the first experimental proof of Maxwell's prediction (without waves): all the currents are closed either geometrically or by the displacement current. It anticipated the experimental evidence put forward by H. Hertz with electromagnetic waves.

A. Eichenwald later made quantitative measurements [105–107]. His setup featured a capacitor with two metallic rings of breadth a cut by a small interspace. A rubber dielectric (permittivity ϵ) was placed between both electrodes. In a first series of experiments, the insulator was fixed while the rings were rotated. In a second series, Eichenwald put into motion altogether the dielectric and the capacitor plates. Each plate carries a surface density of charges $\sigma_p = \epsilon E$ where E is the applied electric field. The dielectric disk has obviously the opposite charge when at rest. Following Rowland's work, one would expect, in the first series of experiments, a convection current $I_R = \sigma_p a v = \epsilon E a v$ when the insulator of breadth a is in motion at constant velocity v. However, Eichenwald measured a lower value $I_{E_1} = (\epsilon - \epsilon_0) E a v = \sigma_i a v$ as if the surface charge of the insulator in motion was $\sigma_i = (\epsilon - \epsilon_0) E$. This polarization charge due to motion would correspond to the static charge at the interface of a metallic conductor created by a reduced effective electric field $E_i = (1 - \epsilon_0/\epsilon) E$ (see [108] where Pauli shows how the introduction of an insulator in a capacitor modifies the surface charges). In the second series of experiments, Eichenwald observed a current which was independent of the dielectric constant of the insulator [105–107]. Indeed, the contributions depending on ϵ of both σ_p and σ_i cancel each other $I_{E_2} = (\sigma_p - \sigma_i) a v = \epsilon E a v - (\epsilon - \epsilon_0) E a v = \epsilon_0 E a v$.

7 Wilson's effect and homopolar induction

The experiment of Wilson demonstrates that a dielectric which moves at a constant speed $v = \omega r$ in an induction field produced a polarization inside itself. All the fields are supposed to be stationary.

Superficial electric charges appear on a dielectric in motion when submitted to a uniform induction field $\mathbf{B} = B\mathbf{e_z}$. H.A. Wilson designed a dielectric with the shape of a hollow cylinder (thickness $\mathrm{d} = R_2 - R_1$) that he put in the gap of a magnet [109–115].

Two setups are described in the literature:

- Either both internal and external sides of the cylinder are coated with metallic conductors connected to an electrometer. The latter has a capacity and both surfaces are charged in opposition.
- Or the dielectric tube rotates between the plates of a condenser which is short-circuited.

If the cylinder was an ohmic conductor, conduction electrons would be driven toward the axis by the "motional" electric field $\mathbf{v} \times \mathbf{B}$. Positive charges would appear on the periphery of the cylinder. Hence, a negative electrostatic volume charge density must balance the surface charge density, the resulting $\mathbf{E}$ cancels the motional field. Seen from the cylinder in motion, the electric field transforms according to the magnetic limit of LBLL: $\mathbf{E}'_m \simeq \mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m = \mathbf{E}_m + vB_m\mathbf{e}_r$. Hence, $\mathbf{E}'_m = \mathbf{0}$.

According to the magnetic limit of Galilean Electromagnetism, an ohmic conductor with the shape of a cylindrical tube rotating in a vertical induction field in the laboratory rest frame is submitted to the potential difference,

$$\varDelta_{\mathrm{Ohm}} V_{12} \simeq -\int_{R_1}^{R_2} v B_m \mathrm{d}r \simeq -\frac{B_m \omega (R_2^2 - R_1^2)}{2}\,. \tag{24}$$

If the gap between the inner and outer faces of the cylinder is small ($d \ll R_1 \simeq R_2 \simeq R$), one would have obtained

$$\varDelta_{\mathrm{Ohm}} V_{12} \simeq -B_m R\omega d, \tag{25}$$

which does correspond to the usual formula for the so-called homopolar induction.

The derivation of Wilson in his paper of 1905 is based on the following formula [109]: $\mathbf{D}_{Wilson} = \epsilon_0\epsilon_r\mathbf{E} + \epsilon_0(\epsilon_r - 1)\mathbf{v}\times\mathbf{B}$ which he justified by the facts that the vacuum displacement is not influenced by the cylinder's motion and only the bounded electrons within the dielectric in motion are submitted to the electromotive force $\mathbf{v}\times\mathbf{B}$.

Then, he assumed that the total displacement is circuital $\nabla\cdot\mathbf{D} = 0$ since there is no free charges. As a consequence, whatever the radius r is, we have the relation $2\pi r L D = -Q$ where L is the length of the cylinder and $-Q$ the charge induced on the inner face of the outer metallic coating because Q denotes the charge induced by the polarization on the outer face of the rotating cylinder.

The setup used by Harold Wilson is such that the coatings are linked to an electrometer [109]. The electric field measured by the electrometer is $\varDelta_{\mathrm{Wilson}} V_{12} = -Q/C_e$ where C_e is the capacity of the electrometer. We find

$$-Q = 2\pi r L\epsilon_0\left[-\epsilon_r\frac{\mathrm{d}V}{\mathrm{d}r} + (\epsilon_r - 1)2\pi f r B\right]. \tag{26}$$

Now, we can integrate in order to derive the difference of potential in the rest frame,

$$\mathrm{d}V = \frac{Q}{2\pi L\epsilon_0\epsilon_r}\frac{\mathrm{d}r}{r} + \left(1 - \frac{1}{\epsilon_r}\right)2\pi f B r\mathrm{d}r, \tag{27}$$

that is

$$V_2 - V_1 = \frac{Q}{2\pi L\epsilon_0\epsilon_r}\,\mathrm{Log}\,\frac{R_2}{R_1} + \left(1 - \frac{1}{\epsilon_r}\right)\pi f B(R_2^2 - R_1^2). \tag{28}$$

The capacity of a dielectric tube is $C_d = 2\pi\epsilon_0\epsilon_r\frac{L}{\mathrm{Log}\,\frac{R_2}{R_1}}$. Hence,

$$\varDelta_{\mathrm{Wilson}} V_{12} = -\frac{Q}{C_e} = \frac{Q}{C_d} + \left(1 - \frac{1}{\epsilon_r}\right)\pi f B(R_2^2 - R_1^2) = \frac{Q}{C_d} - \left(1 - \frac{1}{\epsilon_r}\right)\varDelta_{\mathrm{Ohm}} V_{12}. \tag{29}$$

The control parameter in the experiments is the frequency of rotation ($f = \omega/2\pi$). In the case where ϵ_r is close to unity as for air, the potential difference vanishes as in the first experimental attempts by Blondot before the successful experiments of Wilson (with $\epsilon_r \neq 1$) who showed that the potential difference is a linear function of the frequency of rotation.

8 Wilson and Wilson's effect

8.1 Historical treatment

Soon after the experiment of Wilson, Einstein and Laub in 1908 [71–74] proposed to use a magnetic insulator in order to discriminate between the various theories of electrodynamics in moving media. Then, Marjorie and Harold Wilson created an artificial medium with both magnetic and electric properties by plunging steel balls in a wax forming the cylinder with the same setup as in the experiment of Harold Wilson [116,39,40,117–120,32,44]. Einstein and Laub [71–74] used one of the formula derived in sect. 1,

$$\mathbf{D} = \frac{1}{1-\mu\epsilon v^2}\left[\epsilon\left(1 - \frac{v^2}{c^2}\right)\mathbf{E} + \left(\mu\epsilon - \frac{1}{c^2}\right)\mathbf{v}\times\mathbf{H} - \epsilon\left(\mu\epsilon - \frac{1}{c^2}\right)(\mathbf{v}\cdot\mathbf{E})\mathbf{v}\right]. \tag{30}$$

With the peculiar geometry of the experiment, it becomes

$$D_r = \frac{1}{1-\mu\epsilon v^2}\left[\epsilon\left(1 - \frac{v^2}{c^2}\right)E_r + \left(\mu\epsilon - \frac{1}{c^2}\right)vH_z\right]. \tag{31}$$

Now, if one assume that the capacitor's plates are short-circuited, then $E_r = 0$

$$D_r = \frac{(\mu\epsilon - 1/c^2)}{(1-\mu\epsilon v^2)}vH_z = \frac{(\mu_r\epsilon_r - 1)}{(1-\mu_r\epsilon_r v^2/c^2)}\frac{vH_z}{c^2}. \tag{32}$$

Einstein and Laub considered the following approximation $v \ll c$, then they obtained from the previous relativistic formula the expression [71–75],

$$D_r \simeq (\mu_r \epsilon_r - 1)\frac{vH_z}{c^2}\,. \tag{33}$$

Now, with the additional "Galilean" constitutive relations $H_z = B_z/(\mu_0\mu_r)$ and $D_r = \epsilon_0 E_r + P_r = \sigma$ the surface charge in the laboratory frame, we get a formula which was actually tested in the experiments of Wilson and Wilson using an electrometer,

$$\sigma \simeq (\mu_r\epsilon_r - 1)\frac{\epsilon_0}{\mu_r} vB_z = \epsilon_0 \frac{(\mu_r\epsilon_r - 1)}{\mu_r} vB_z. \tag{34}$$

This Einstein and Laub derivation is straightforward but has no real physical insights. Its major drawback is that it mixes a relativistic formula and Galilean ones. Experimentally, the factor $(\epsilon_r - 1/\mu_r)$ was recovered and was considered as a major argument for the relativity theory against the previous theory of Lorentz which predicted only a $(\epsilon_r - 1)$ factor (tested in Wilson's first experiment and which discarded, for example, Hertz's theory) since it did not take into account the effect of magnetization that is as if $\mu_r = 1$.

8.2 Galilean treatment

The Galilean treatment is now obvious since the Wilson and Wilson's experiment is one example of the superposition derived in sect. 4 (eq. (19)) for the simple case $\mathbf{E}_m = \mathbf{0}$,

$$\mathbf{D}_{\text{Wilson}} \simeq \epsilon\mathbf{E}_e + \left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times\mathbf{B}_m. \tag{35}$$

One has $\mathbf{D}_{\text{Wilson}} \neq 0$ when the capacitor plates are not short-circuited. One replaces the formula used by Wilson ($\mathbf{D}_{\text{Wilson}} = \epsilon\mathbf{E} + (\epsilon - \epsilon_0)\mathbf{v}\times\mathbf{B}$) by the last one and we can calculate accordingly the potential difference taking into account the influence of the relative permeabilitty in the Wilson and Wilson's experiment.

Similarly to the Wilson effect, we get

$$\Delta_{\text{Wilsons}} V_{12} = -\frac{Q}{C_e} = \frac{Q}{C_d} + \left(1 - \frac{1}{\mu_r\epsilon_r}\right)\pi f B(R_2^2 - R_1^2) = \frac{Q}{C_d} - \left(1 - \frac{1}{\mu_r\epsilon_r}\right)\Delta_{\text{Ohm}} V_{12}. \tag{36}$$

Otherwise, one has directly

$$\epsilon\mathbf{E}_e + \left(\epsilon - \frac{1}{\mu c^2}\right)\mathbf{v}\times\mathbf{B}_m \simeq \mathbf{0}, \tag{37}$$

when the electrometer is not used. With $\mathbf{E}_e = -\nabla V = -(1 - \frac{1}{\epsilon_r\mu_r})\mathbf{v}\times\mathbf{B}_m$, one finds the observed voltage on sliding contacts,

$$\Delta_{\text{Wilson}} V_{12} = \left(1 - \frac{1}{\mu_r\epsilon_r}\right)\Delta_{\text{Ohm}} V_{12} \simeq -\left(1 - \frac{1}{\mu_r\epsilon_r}\right)\pi f B(R_2^2 - R_1^2). \tag{38}$$

The measurements by the Wilsons and their modern reproduction by Hertzberg *et al.* displayed unambiguously a linear relationship between the voltage and the frequency as well as the factor $1 - 1/(\epsilon_r\mu_r)$ [116,39,40,117–120,32,44]. However, what this experiment validates is first of all the Galilean Electrodynamics *à la* Minkowski. The Special Relativity prediction was not tested so far contrary to what was/is believed and is unlikely to be because of the rapid velocities it implies...

Concluding remarks

One century after the seminal work of Minkowski, the electrodynamics of moving continuous media is still a subject of investigations for research and should be included in physics lectures as early as possible. As a conclusion, Minkowski's electrodynamics is *useless* when one deals with *low velocities*. However, *only* the Maxwell-Minkowski equations are able to predict correctly the optics of moving media like the Cerenkov radiation [121] or the Fresnel-Fizeau drag [122]. The Galilean limits provide an efficient way to analyse a large amount of phenomena studied by both engineers and physicists.

We applied the taking of limit *à la* Lévy-Leblond to the constitutive relations introduced by Minkowski in 1908 to explain all the experiments of electrodynamics of moving bodies. Indeed, the continuous polarized media are described by a functional relation between fields on one hand and inductions on the other hand. Usually, they are linear via coefficients of proportionality (permittivity for insulators, permeability for magnets) but are valid only at rest. Minkowski made the hypothesis that the form in the moving frame of reference of the constitutive relations was the same as in the rest and that to obtain their expression in the laboratory frame, it was necessary to apply the transformations of Lorentz to the fields and their inductions. The constitutive relations become functions of the relative speed, the lengths factor of the FitzGerald-Lorentz contraction and bring in crossed terms. For example, the magnetic induction in the laboratory frame expresses itself not only with the magnetic field but also with the electric field of the same frame. In the past, the physicists were able to interpret all the experiments of electrodynamics of moving bodies with the relativist constitutive Minkowski relations. However, the presence of the factor of contraction is an indication that the theory of Minkowski is particularly adapted to high speed and thus to experiments of optics of moving bodies like the Fresnel-Fizeau's effect or to experiments of electrodynamics of moving bodies like the Vavylov-Cherenkov's effect with fast particles where the phenomena inherent to Special Relativity are obvious (contraction of the lengths and the dilation of time). For the experiments of electrodynamics of moving bodies with low speeds, the Galilean theory is the most adapted because it is easier of stake in work from the calculus point of view and does not bring in the kinematics effect of Special Relativity which are absolutely unimportant in the Galilean limit. In conclusion, the Galilean Electromagnetism discovered after the Relativistic Electromagnetism seems to have to take its place next to the Newton's Mechanics. These two theories have, naturally, a domain of limited validity, but stay nevertheless, very useful in the practical explanation of the Galilean phenomena. Let us remind the prediction of Poincaré on the future of Newton's theory after the invention of Special Relativity: "Today certain physicists want to adopt a new convention... Those who are not of this opinion can keep the former in order not to disturb their old customs. I believe, between us, that it is what they will make even for a long time". Let us wish that the Galilean electromagnetism replaces Special Relativity in the common practice...

A long time ago, Laue then Pauli then Arzeliès lamented for the fact that the generalized Roentgen-Einchenwald's effect (dual to the Wilson and Wilson effect) taking into account the factor $(\mu\epsilon - 1/c^2)$ has never been performed so far (we leave the derivation to the reader starting with the constitutive relations $\mathbf{H}_e \simeq \frac{\mathbf{B}_e}{\mu} + (\epsilon - \frac{1}{\mu c^2})\frac{\mathbf{v}\times\mathbf{E}_e}{c^2}$ and $\mathbf{D}_e \simeq \epsilon\mathbf{E}_e$): some experimental works are needed to complete our understanding of moving polarized media. We plan to come back on the galilean limits of constitutive relations by focusing on the mathematical duality between both limits as described by Ridgely for the relativistic case [41,42]...

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