Eur. J. Phys. **28** (2007) L7–L9 doi:10.1088/0143-0807/28/3/N01

LETTERS AND COMMENTS

Bead, hoop and spring . . . : some theoretical remarks

Germain Rousseaux

Université de Nice-Sophia Antipolis, Institut Non-Linéaire de Nice, UMR 6618 CNRS-UNICE, 1361, route des Lucioles, 06560 Valbonne, France

E-mail: Germain.Rousseaux@inln.cnrs.fr

Received 5 September 2006, in final form 14 February 2007
Published 7 March 2007
Online at stacks.iop.org/EJP/28/L7

Abstract
Here, we solve a simplified version (corresponding to oscillations with small amplitudes) of the nonlinear equation describing the evolution of the bead, hoop and spring problem derived by Ochoa and Clavijo (2006 *Eur. J. Phys.* **27** 1277–88) with the so-called amplitude equation method. We point out the analogy with the equation derived by us previously in Rousseaux *et al* (2005 *Eur. J. Phys.* **26** 1065–78) and which describes the nonlinear dynamics of a conical pendulum. Our comment illustrates the usefulness of nonlinear techniques for teachers, such as the amplitude equation formalism, since these can be applied to all nonlinear oscillators.

In [1], the authors have studied a mechanical system made of two beads of mass m linked by a spring which is bound to slide on a horizontal loop of radius R. The latter is put into rotation with pulsation ω around a horizontal axis denoted by z. The centrifugal force tends to extend the distance between the beads, whereas the elasticity of the spring k counteracts this effect. Each variation of the distance $2r$ between both beads whose initial value is $2r_0$ translates into a changing position of the centre of mass of the two beads along the horizontal axis. Obviously, there are two geometrical constraints $r_0 < R$ and $r = \pm\sqrt{R^2 - z^2}$.

Ochoa and Clavijo have derived the Lagrangian of the system [1]:

$$L = m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - 2k(r - r_0)^2. \tag{1}$$

Then, using the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0, \tag{2}$$

they ended up with the following equation of motion:

$$\ddot{z} + \left(\frac{z}{R^2 - z^2}\right)\dot{z}^2 + \left(\frac{2k}{mR^2} - \frac{\omega^2}{R^2}\right)z^3 + \frac{2kr_0}{mR^2}z\sqrt{R^2 - z^2} - \left(\frac{2k}{m} - \omega^2\right)z = 0, \tag{3}$$

and they commented on its form being 'quite complex to solve'. The purpose of this comment is to display its solutions corresponding to oscillations. Moreover, we point out the similarity with the equation that we derived in [2] which describes the evolution of a conical pendulum.

Our derivation is a pedagogic example of the way to deal with nonlinear oscillators which are ubiquitous in lectures on classical mechanics. We introduce the techniques of the so-called amplitude equation which can be seen as the nonlinear generalization of the normal modes approach for a linear oscillator.

We use dimensionless variables $z = Z \times R$ and suppose for the moment that $\omega^2 > \frac{2k}{m}$, otherwise the equation will not have the same form (with a positive sign in front of Z) as the one for a simple oscillator: $\ddot{Z} + \left(\omega^2 - \frac{2k}{m}\right)Z =$ other terms.

Let us denote $\omega_0^2 = \omega^2 - \frac{2k}{m}$. The evolution equation becomes

$$\ddot{Z} + \left(\frac{Z}{1 - Z^2}\right)\dot{Z}^2 - \omega_0^2 Z^3 + \frac{2kr_0}{mR} Z\sqrt{1 - Z^2} + \omega_0^2 Z = 0. \tag{4}$$

Now, we assume that the displacements are small, $Z \ll 1$, and we introduce $\omega_1^2 = \frac{2kr_0}{mR}$ to get

$$\ddot{Z} + \left(\omega_0^2 + \omega_1^2\right)Z - \left(\omega_0^2 + \frac{\omega_1^2}{2}\right)Z^3 + Z\dot{Z}^2 \simeq 0. \tag{5}$$

Ochoa and Clavijo have introduced a peculiar pulsation ω_c such that $\omega_c^2 = \frac{2k}{m}\left(1 - \frac{r_0}{R}\right)$ and which allows us to interpret the bead, hoop and spring problem as a critical phenomenon in analogy with statistical mechanics [1].

We introduce additional parameters a and b in order to simplify the resolution ($a^2 = \omega_0^2 + \omega_1^2 = \omega^2 - \omega_c^2$ and $b^2 = \omega_0^2 + \frac{\omega_1^2}{2}$):

$$\ddot{Z} + a^2 Z - b^2 Z^3 + Z\dot{Z}^2 = 0. \tag{6}$$

At this stage, we can relax the stronger constraint $\omega^2 > \frac{2k}{m}$ for a weaker one which will still correspond to oscillations. $a^2 > 0$ implies $\omega^2 > \frac{2k}{m}\left(1 - \frac{r_0}{R}\right) = \omega_c^2$: we do recover the critical pulsation of Ochoa and Clavijo.

With $\tau = at$ ($\partial_\tau = {}'$) and $0 < \alpha = \frac{b^2}{a^2}$, the equation of motion takes the simplest form

$$Z'' + Z - \alpha Z^3 + ZZ'^2 = 0. \tag{7}$$

In [2], we derived the equation of motion for a vertical conical pendulum whose projections in a horizontal plane (X, Y) are described with the complex variable $W = X + \mathrm{i}Y$:

$$W'' + W - \tfrac{1}{2}|W|^2 W + W|W'|^2 = 0. \tag{8}$$

Here, $|W|^2 = W\overline{W}$ where the bar denotes the complex conjugate (c.c.).

The cubic term without the time derivative comes from the nonlinearity associated with the motion of the pendulum on the peculiar geometry of the sphere, whereas the other cubic term with the time derivative stands for the centrifugal force effect.

We now use the same method based on the formalism of the so-called amplitude equation as in [2] with $Z = A\,\mathrm{e}^{\mathrm{i}\tau} + \text{c.c.}$ (let us recall that Z is real, A is complex and varies slowly with time), and we find the usual form corresponding to a nonlinear oscillator:

$$A' = \frac{1 - 3\alpha}{2}\mathrm{i}|A|^2 A. \tag{9}$$

Indeed, for slow changes of the amplitude with time ($A' \ll A$), we have used the approximation

$$Z' \simeq \mathrm{i}(A\exp(\mathrm{i}\tau) - \overline{A}\exp(-\mathrm{i}\tau)) \tag{10}$$

in order to get

$$Z'' + Z \simeq 2\mathrm{i}(A' \exp(\mathrm{i}\tau) - \overline{A'} \exp(-\mathrm{i}\tau)), \tag{11}$$

where the slowly varying envelope approximation was introduced: $A'' \ll A'$.

$ZZ'^2 = |A|^2 A\,\mathrm{e}^{\mathrm{i}\tau} + \mathrm{c.c.}$ is the contribution of the centrifugal force where we kept only the resonant term oscillating like $\mathrm{e}^{\mathrm{i}\tau}$: one says that the harmonics were 'time averaged' with respect to this so-called secular term. The cubic term was treated accordingly: we kept only the resonant term in its development, $A^3 \exp(3\mathrm{i}\tau) + 3A^2\overline{A} \exp(\mathrm{i}\tau) + 3A\overline{A}^2 \exp(-\mathrm{i}\tau) + \overline{A}^3 \exp(-3\mathrm{i}\tau)$. A Poincaré–Lindstedt expansion or the normal form theory would have led to the same equation [3].

Now, we can compare our result with the amplitude equation for the simplest nonlinear pendulum [2, 3]:

$$A' = -\frac{\mathrm{i}}{4}|A|^2 A. \tag{12}$$

Both feature invariance by rotation in the complex plane which is associated with the translation in time: $A \to A\,\mathrm{e}^{\mathrm{i}\phi}$. Moreover, the energy is conserved as $\partial_\tau(|A|^2) = 0$: Ochoa and Clavijo have illustrated this last point in their figure 8 where they plotted in fact the phase space of a nonlinear oscillator whose period depends on the initial amplitude [1].

We would like to point out that, depending on the value of α, the nonlinearity of the pendulum of Ochoa and Clavijo can be positive ($1/3 > \alpha$), negative ($1/3 < \alpha$) or null ($1/3 = \alpha$) compared to the usual nonlinear pendulum whose period increases with the amplitude as the coefficient of $|A|^2 A$ is negative ($-1/4 < 0$).

As a final remark, the amplitude equation we derived can be seen as the analogue of the so-called Landau equation in statistical physics and it underlines in the clearest manner the analogy pointed out by Ochoa and Clavijo [1] between the mechanical problem we have treated and a critical phenomenon: the Lagrangian plays the role of a free energy 'à la Landau' which is minimized in order to get the Landau equation. We hope that both students and teachers of physics will find the analogy useful for tackling other phase transitions from the mechanical point of view and conversely as discussed, for example, in [4].

References

[1] Ochoa F and Clavijo J 2006 Bead, hoop and spring as a classical spontaneous symmetry breaking problem *Eur. J. Phys.* **27** 1277–88

[2] Rousseaux G, Coullet P and Gilli J M 2005 Amplitude equations for mechanical analogues of Faraday and nonlinear optical rotations *Eur. J. Phys.* **26** 1065–78

[3] Nayfeh A H 1993 *Method of Normal Forms* (New York: Wiley)

[4] Sivardière J 1997 Simple mechanical systems exhibiting instabilities *Eur. J. Phys.* **18** 384–7