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On the electrodynamics of moving bodies at low velocities

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Abstract

We discuss an article by Le Bellac and Lévy-Leblond in which they have identified two Galilean limits of electromagnetism (1973 *Nuovo Cimento* B **14** 217–33). We use their results to point out some confusion in the literature, and in the teaching of special relativity and electromagnetism. For instance, it is not widely recognized that there exist *two* well-defined non-relativistic limits, so that researchers and teachers are likely to utilize an incoherent mixture of both. Recent works have shed new light on the choice of gauge conditions in classical electromagnetism. We retrieve the results of Le Bellac and Lévy-Leblond first by examining orders of magnitudes and then with a Lorentz-like manifestly covariant approach to Galilean covariance based on a five-dimensional Minkowski manifold. We emphasize the Riemann–Lorenz approach based on the vector and scalar potentials as opposed to the Heaviside–Hertz formulation in terms of electromagnetic fields.

1. Introduction

Although special relativity has superseded Galilean relativity as an appropriate framework to describe high-energy phenomena, there exists a wealth of low-energy systems, such as in condensed matter physics and low-energy nuclear physics, where the usefulness of Galilean covariance should be better understood and appreciated. The main purpose of this paper is to emphasize the relevance of Galilean covariance nowadays, nearly 100 years after Lorentz, Poincaré and Einstein developed a theory that turned into special relativity, in order to mend the (at times) apparent incompatibility between Galilean mechanics and the full set of Maxwell equations [2]. The title is reminiscent of Einstein's famous paper [3] because we wish to point out that, had the existence and structure of two Galilean limits of electromagnetism been properly recognized at the end of the nineteenth century, some phenomena could have been understood without now referring to them as 'relativistic phenomena' (the very concept of spin is an example [4]). Indeed, almost 70 years after Einstein's article, Le Bellac and Lévy-Leblond (LBLL) observed that there exist not only one, but *two* well-defined Galilean (that is, non-relativistic) limits of electromagnetism: the so-called magnetic and electric limits [1].

We wish to point out hereafter some confusion, in the research literature and in physics education, which results from an incoherent mixture of the two Galilean limits of electromagnetism. This follows from inaccurate definitions of non-relativistic covariance, which is why we emphasize at once that the definition of Galilean covariance employed henceforth in this paper rests on its compatibility with the Galilean transformations of spacetime (equation (6)). Examples of such misleading, though well known, text presentations are mentioned in [1], and there have been more since then. The fact that one should be careful when dealing with electrodynamics at low velocities has been illustrated, for instance, in [5]. Let us illustrate this point with a simple example. Under a Lorentz transformation with relative velocity $\mathbf{v}$, the electric and magnetic fields, in vacuum, become

$$\begin{aligned}\mathbf{E}' &= \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{\mathbf{v}^2},\\ \mathbf{B}' &= \gamma\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{B})}{\mathbf{v}^2},\end{aligned} \tag{1}$$

respectively. The fact that Galilean covariance is a much more subtle concept than simply taking the $v \ll c$, or $\gamma \simeq 1$, limit is illustrated by the fact that equation (1) then becomes

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B}, \qquad \mathbf{B}' = \mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}, \tag{2}$$

which not only is incompatible with Galilean relativity, but does not even satisfy the composition properties of transformation groups [1, 5]. That is to say, a sequence of such transformations does not have the same form as above.

We have organized this paper as follows. In section 2, we recall the main results of LBLL [1]. In section 3, we obtain these results using two arguments: one based on orders of magnitudes and a recent covariant approach with which the Galilean spacetime is embedded into a five-dimensional space. The purpose of this second approach is to allow the use of relativistic methods to solve non-relativistic problems. Likewise, the tensor methods introduced in the teaching of special relativity may now be applied in a Galilean context. Indeed, the calculations in section 3.2 should be accessible to many advanced undergraduate physics students. Throughout the paper, we favour the Riemann–Lorenz formulation of electrodynamics, based on the scalar and vector potentials, over the Heaviside–Hertz approach which involves electromagnetic fields [4]. Discussion and applications are in section 4.

2. Galilean electromagnetism

The purpose of LBLL was to write the laws of electromagnetism in a form compatible with Galilean covariance rather than Lorentz covariance. As LBLL put it, such laws could have been devised by a physicist in the mid-nineteenth century [1]. Here, let us retrieve these laws

[4] In this paper, we emphasize the Riemann–Lorenz approach to electromagnetism. Therein the central role is played by the vector and scalar potentials, unlike the Heaviside–Hertz approach, which rather relies on the fields themselves. For a justification, see [13].

from relativistic kinematics. The Lorentz transformation of a 4-vector $(u^0, \mathbf{u})$, where the four components have the same units, is given by (see, e.g., chapter 7 of [6])

$$u'^0 = \gamma\left(u^0 - \frac{1}{c}\mathbf{v}\cdot\mathbf{u}\right), \qquad \mathbf{u}' = \mathbf{u} - \gamma\frac{\mathbf{v}}{c}u^0 + (\gamma - 1)\frac{\mathbf{v}}{\mathbf{v}^2}\mathbf{v}\cdot\mathbf{u}, \tag{3}$$

where $\gamma \equiv \frac{1}{\sqrt{1-\mathbf{v}^2/c^2}}$, with a relative velocity $\mathbf{v}$. The speed of light in the vacuum is denoted by c. LBLL were the first to observe that this transformation admits two well-defined Galilean limits [1]. One limit is for timelike vectors:

$$u'^0 = u^0, \qquad \mathbf{u}' = \mathbf{u} - \frac{1}{c}\mathbf{v}u^0, \tag{4}$$

which, as we shall see, may be related to the so-called *electric* limit. The second limit is for spacelike vectors:

$$u'^0 = u^0 - \frac{1}{c}\mathbf{v}\cdot\mathbf{u}, \qquad \mathbf{u}' = \mathbf{u}, \tag{5}$$

and will be associated with the *magnetic* limit. As it is well known, the spacetime coordinates can be described by timelike vectors only. Indeed, equation (4) has the form of Galilean inertial spacetime transformations:

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \qquad t' = t. \tag{6}$$

Nevertheless, other vectors, such as the 4-potential and 4-current, may transform as one or other of the two limits.

An example of the subtlety of non-relativistic kinematical covariance is that it is quite common to neglect to enforce the condition that a non-relativistic limit involves not only low-velocity phenomena, but also large timelike intervals so one obtains different kinematics, referred to as Carroll kinematics [7]. In other terms, a Galilean world is one within which units of time are naturally much larger than units of space. The existence of events physically connected by large spacelike intervals would imply loss of causality, among other things. Other such kinematics, each one being some limit of the de Sitter kinematics, have been classified in [8].

The situation is similar to electric and magnetic fields. One needs to compare the module of the electric field E to c times the module of the magnetic field, i.e. cB. When the magnetic field is dominant, equation (1) reduces to a transformation referred to as the *magnetic limit* of electromagnetism:

$$\begin{aligned} \mathbf{E}'_m &= \mathbf{E}_m + \mathbf{v}\times\mathbf{B}_m, \qquad E_m \ll cB_m, \\ \mathbf{B}'_m &= \mathbf{B}_m. \end{aligned} \tag{7}$$

The other alternative, where the electric field is dominant, leads to the *electric limit*:

$$\begin{aligned} \mathbf{E}'_e &= \mathbf{E}_e, \qquad E_e \gg cB_e, \\ \mathbf{B}'_e &= \mathbf{B}_e - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}_e. \end{aligned} \tag{8}$$

Indeed, the approximations $E_e/c \gg B_e$ and $v \ll c$ together imply that $E_e/v \gg E_e/c \gg B_e$ so that we take $E_e \gg vB_e$ in equation (1). Such an analysis of orders of magnitude is described in the following section.

From the Galilean transformations of spacetime, equation (6), we find

$$\nabla' = \nabla, \qquad \partial_{t'} = \partial_t + \mathbf{v}\cdot\nabla. \tag{9}$$

The fields' transformations in the *magnetic* limit of equation (7) are clearly compatible with the use of equation (9) together with the transformations of the 4-potential $(V, \mathbf{A})$:

$$V'_m = V_m - \mathbf{v} \cdot \mathbf{A}_m, \qquad \mathbf{A}'_m = \mathbf{A}_m, \tag{10}$$

(note the similarity with equation (5)) where

$$\mathbf{E}_m = -\nabla V_m - \partial_t \mathbf{A}_m, \qquad \mathbf{B}_m = \nabla \times \mathbf{A}_m. \tag{11}$$

Similarly, the *electric* limit of equation (8) may be obtained from equation (9) and the transformations of the 4-potential:

$$V'_e = V_e, \qquad \mathbf{A}'_e = \mathbf{A}_e - \frac{\mathbf{v}}{c^2} V_e. \tag{12}$$

This equation is similar to equation (4). Now, however, that the fields are related to the 4-potential by

$$\mathbf{E}_e = -\nabla V_e, \qquad \mathbf{B}_e = \nabla \times \mathbf{A}_e. \tag{13}$$

In parallel with the two possible sets of transformations of the 4-potential, there are two ways to transform the 4-current $(\rho, \mathbf{j})$. In the *magnetic limit*, it transforms as equation (5),

$$\rho'_m = \rho_m - \frac{1}{c^2} \mathbf{v} \cdot \mathbf{j}_m, \qquad \mathbf{j}'_m = \mathbf{j}_m, \tag{14}$$

and the continuity equation then reads

$$\nabla \cdot \mathbf{j}_m = 0. \tag{15}$$

The appearance of an 'effective' charge density $\rho'_m = \rho_m - \frac{1}{c^2}\mathbf{v} \cdot \mathbf{j}_m$ is certainly one of the salient features of the magnetic limit. We will refer the interested reader to the works in [9], which discuss the effect of this effective charge without pointing out its Galilean origin for most of them.

For the *electric limit*, it transforms as equation (4),

$$\rho'_e = \rho_e, \qquad \mathbf{j}'_e = \mathbf{j}_e - \mathbf{v}\rho_e, \tag{16}$$

and the continuity equation has its usual form

$$\nabla \cdot \mathbf{j}_e + \partial_t \rho_e = 0. \tag{17}$$

Finally, Maxwell's equations,

$$\begin{aligned}
&\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, && \text{Faraday,} \\
&\nabla \cdot \mathbf{B} = 0, && \text{Thomson,} \\
&\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \partial_t \mathbf{E}, && \text{Ampère,} \\
&\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho, && \text{Gauss,}
\end{aligned} \tag{18}$$

reduce, in the Galilean limits, to two respective forms. As the field transformation laws (1) themselves, this fact is not so obvious if one naively takes the limit $c \to \infty$. In the following section, we present an argument based on dimensional analysis and orders of magnitude. In [1], it was found that, in the electric limit, the Maxwell equations reduce to

$$\begin{aligned}
&\nabla \times \mathbf{E}_e = \mathbf{0}, && \nabla \cdot \mathbf{B}_e = 0, \\
&\nabla \times \mathbf{B}_e - \frac{1}{c^2} \partial_t \mathbf{E}_e = \mu_0 \mathbf{j}_e, && \nabla \cdot \mathbf{E}_e = \frac{1}{\epsilon_0} \rho_e.
\end{aligned} \tag{19}$$

Clearly, the main difference to the relativistic Maxwell equations is that here the electric field has zero curl in Faraday's law. In the magnetic limit, the Maxwell equations become

$$\begin{aligned}
&\nabla \times \mathbf{E}_m = -\partial_t \mathbf{B}_m, && \nabla \cdot \mathbf{B}_m = 0, \\
&\nabla \times \mathbf{B}_m = \mu_0 \mathbf{j}_m, && \nabla \cdot \mathbf{E}_m = \frac{1}{\epsilon_0} \rho_m.
\end{aligned} \tag{20}$$

The displacement current term is absent in Ampère's law.

3. Recent analyses

3.1. Orders of magnitude

Many errors occur within low-velocity limits of relativistic theories when one naively replaces some quantities with zero, rather than carefully comparing various orders of magnitudes involved in the equations. As we shall show hereafter, we do not require sophisticated mathematical tools to retrieve the two Galilean limits of electromagnetism of LBLL. As discussed by one of us in [10], a careful dimensional analysis of the fields' equations is sufficient for this purpose. Therein, it is argued that the electric and magnetic limits may be retrieved by a careful consideration of the order of magnitude of the dimensionless parameters,

$$\varepsilon \equiv \frac{L}{cT} \qquad \text{and} \qquad \xi \equiv \frac{j}{c\rho}, \tag{21}$$

where L, T, j and ρ represent the orders of magnitude of length, time, current density and charge density, respectively. The Galilean kinematics considered hereafter corresponds to the quasistatic limit $\varepsilon \ll 1$.

The electric or magnetic character of the Galilean limits of electromagnetism is determined by the behaviour of the parameter ξ. From Gauss's law and Ampère's law, equation (18), we find $\frac{cB}{E} \simeq \frac{j}{\rho c}$, so that

$$\frac{cB}{E} = \xi.$$

Using this result and equations (7), (8), we find

$$\begin{aligned} &\xi \gg 1: \quad \text{magnetic limit,} \\ &\xi \ll 1: \quad \text{electric limit.} \end{aligned} \tag{22}$$

Returning to equation (21), we see that the magnetic limit corresponds to the approximation $j \gg c\rho$, that is, the spacelike component is larger than the timelike component. This echoes the transformation in equation (5). Conversely, the electric limit corresponds to the approximation $c\rho \gg j$, so that the spacelike component now is much larger than the timelike component. This is analogous to equation (4).

From the Maxwell displacement current term in Ampère's law, equation (18), we find

$$B \simeq \frac{vE}{c^2}, \tag{23}$$

where v denotes the ratio of orders of magnitude L/T. Similarly, the magnetic induction term of Faraday's law gives

$$E \simeq vB. \tag{24}$$

If we substitute this result into equation (23), we find that the displacement current term and the full Faraday law are compatible only if $v \simeq c$, that is, in the Lorentz covariant regime. However, equation (23) cannot be obtained if we drop $\partial_t \mathbf{E}$ from Ampère's law, so that it is compatible with the first and third equations of (19), i.e. in the electric limit. On the other hand, equation (24) is compatible with the first and third equations of (20), i.e. the magnetic limit, because it does not appear if we drop the magnetic induction term $\partial_t \mathbf{B}$ of Faraday's law in line three of equation (18).

Following the lines of reasoning of [11–15], we use the Riemann–Lorenz formulation of electromagnetism, which relies on the potentials as the basic quantities, in order to retrieve the two Galilean limits. This is in opposition to the Heaviside–Hertz formulation, which is based on the magnetic and electric fields [11–13]. In terms of potentials, the equations of classical electromagnetism read

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \qquad \text{Riemann equations,}$$
$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}, \tag{25}$$

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0, \qquad \text{Lorenz equation,} \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v} + q\mathbf{A}) = -q\nabla(V - \mathbf{v}\cdot\mathbf{A}), \qquad \text{Lorentz force.} \tag{27}$$

The quasistatic approximation, $\varepsilon \ll 1$, of equation (25) reads

$$\nabla^2 V \simeq -\frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla^2 \mathbf{A} \simeq -\mu_0 \mathbf{j}, \tag{28}$$

from which we can define a further dimensionless ratio, $\frac{cA}{V} \simeq \frac{j}{\rho c}$, so that

$$\frac{cA}{V} \simeq \xi. \tag{29}$$

Once again, this echoes our comment following equation (22): in the magnetic limit, the spacelike quantity cA is dominant, whereas in the electric limit, it is the timelike quantity V which dominates.

If we compare the two terms of the Lorenz (not Lorentz [14]) gauge condition, equation (26), we find

$$\frac{|\nabla \cdot \mathbf{A}|}{\partial_t V/c^2} \simeq \frac{cT}{L}\frac{cA}{V} \simeq \frac{\xi}{\varepsilon}. \tag{30}$$

In the quasistatic regime, $\varepsilon \ll 1$, we find therefore two possibilities. If $\xi \ll 1$, like ε, we are in the electric limit, and the gauge condition is similar to the Lorenz condition:

$$\nabla \cdot \mathbf{A}_e + \frac{1}{c^2}\partial_t V_e = 0. \tag{31}$$

On the other hand, in the magnetic limit, $\xi \gg 1$, we drop $\partial_t V$, so that we obtain the Coulomb gauge condition:

$$\nabla \cdot \mathbf{A}_m = 0. \tag{32}$$

Let us use the orders of magnitude for the 4-potential components and obtain thereof their Galilean transformations in the magnetic limit, equation (10), and the electric limit, equation (12). From equation (3) with $u^0 = V/c$ and $\mathbf{u} = \mathbf{A}$, we find that the scalar potential V and the vector potential $\mathbf{A}$ transform, under a Lorentz transformation, as

$$V' = \gamma(V - \mathbf{v}\cdot\mathbf{A}), \qquad \mathbf{A}' = \mathbf{A} - \gamma\frac{\mathbf{v}}{c^2}V + (\gamma - 1)\frac{\mathbf{v}}{\mathbf{v}^2}\mathbf{v}\cdot\mathbf{A}. \tag{33}$$

From the first equation, we have $V \simeq vA$, so that we obtain, from equation (29),

$$\xi = \frac{cA}{V} \simeq \frac{cA}{vA} = \frac{1}{\varepsilon}.$$

Therefore, in the quasistatic limit $\varepsilon \ll 1$, this equation gives $\xi \gg 1$, so that the first line is compatible with the magnetic limit. Accordingly, this is incompatible with the electric limit, for which $\xi \ll 1$, so that the term $\mathbf{v}\cdot\mathbf{A}$ must be dropped from the first line of equation (33) in the electric limit.

A similar argument, applied to the second equation of (33), implies that $A \simeq \frac{vV}{c^2}$, so that

$$\xi = \frac{cA}{V} \simeq \frac{cvV}{c^2 V} = \frac{v}{c} = \varepsilon.$$

Unlike the previous case, the quasistatic limit leads to $\xi \ll 1$, which is compatible with the electric limit only, equation (12). This implies that, in the magnetic limit, the term $\frac{\mathbf{v}}{c^2}V$ must be dropped from the second line of equation (33), as is the case in equation (10).

If we use an entirely similar analysis for the Lorentz transformation of charge and current densities, obtained from equation (3) with $u^0 = \rho$ and $\mathbf{u} = \mathbf{j}/c$,

$$\rho' = \gamma \left(\rho - \frac{1}{c^2} \mathbf{v} \cdot \mathbf{j} \right), \qquad \mathbf{j}' = \mathbf{j} - \gamma \mathbf{v} \rho + (\gamma - 1) \frac{\mathbf{v}}{\mathbf{v}^2} \mathbf{v} \cdot \mathbf{j}, \tag{34}$$

we retrieve the Galilean transformations, equations (14) and (16), for the magnetic and electric limits, respectively.

Let us conclude by briefly discussing the continuity equation:

$$\nabla \cdot \mathbf{j} + \partial_t \rho = 0.$$

If we compare the two terms as we have done for the Lorentz gauge condition in equation (30), we find

$$\frac{|\nabla \cdot \mathbf{j}|}{\partial_t \rho} \simeq \frac{cT}{L} \frac{j}{c\rho} \simeq \frac{\xi}{\varepsilon}.$$

If $\xi \ll 1$, like ε in the quasistatic regime $\varepsilon \ll 1$, then we obtain the electric limit and we retrieve equation (17). On the other hand, in the magnetic case, $\xi \gg 1$, so that we drop $\partial_t \rho$ and thereby obtain equation (15).

3.2. Reduction from (4,1) Minkowski spacetime

Hereafter, we briefly review a different approach to the Galilean gauge fields [16]. It involves a formulation of Galilean invariance based on a reduction from a five-dimensional Minkowski manifold to the Newtonian spacetime [17–19]. It may be of interest to physics teachers because it relies on tensor calculus similar to the one utilized in the teaching of special relativity. We also wish to bring to the attention of physics teachers an interesting advocacy by Kapuścik [19] in favour of working with five dimensions. Therein, Kapuścik provides compelling arguments based on concepts similar to those utilized in the teaching of special relativity, such as clocks, rods, trains and mirrors. Then, the main difference between special relativity and Galilean relativity is the absence, for the latter, of an invariant signal, thereby requiring a ‘control parameter’ that keeps track of the reference frame. This parameter is related to the additional coordinate.

The extended space is such that a Galilean boost with relative velocity $\mathbf{v} = (v_1, v_2, v_3)$ acts on a *Galilei-vector* $(\mathbf{x}, t, s)$ as

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \qquad t' = t, \qquad s' = s - \mathbf{v} \cdot \mathbf{x} + \tfrac{1}{2}\mathbf{v}^2 t. \tag{35}$$

Since ∂_s transforms like the mass m (see below for a justification), one can see the additional coordinate s as being conjugate to the mass m since both are invariant under Galilean transformations. The parameter s may be seen also as the action per unit mass. More about classical and quantum physical interpretations of s is in [16–19].

The scalar product,

$$(A|B) = A^\mu B_\mu \equiv \mathbf{A} \cdot \mathbf{B} - A_4 B_5 - A_5 B_4,$$

of two Galilei-vectors A and B is invariant under the transformation, equation (35). This suggests a method to base the tensor calculus on the metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{36}$$

Hereafter, we refer to this as the *Galilean metric*.

The transformation in equation (35) can be written in matrix form for the components of any 5-vector as

$$x'^{\mu} = \Lambda^{\mu}\nu x^{\nu},$$

where μ denotes the row and ν the column (so that $\Lambda^{\mu}\nu$ is the $(\mu\nu)$-entry) or

$$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \\ x'^4 \\ x'^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -v_1 & 0 \\ 0 & 1 & 0 & -v_2 & 0 \\ 0 & 0 & 1 & -v_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -v_1 & -v_2 & -v_3 & \frac{1}{2}\mathbf{v}^2 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}.$$

For a 5-oneform, this transformation reads

$$x'_{\mu} = \Lambda_{\mu}\nu x_{\nu},$$

where μ now denotes the column and ν the row (that is Λ_{μ}^{ν} is the $(\nu\mu)$-entry) or

$$\left(x'_1, x'_2, x'_3, x'_4, x'_5\right) = (x_1, x_2, x_3, x_4, x_5) \begin{pmatrix} 1 & 0 & 0 & v_1 & 0 \\ 0 & 1 & 0 & v_2 & 0 \\ 0 & 0 & 1 & v_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v_1 & v_2 & v_3 & \frac{1}{2}\mathbf{v}^2 & 1 \end{pmatrix}. \tag{37}$$

We write the embedding as

$$(\mathbf{x}, t) \to x^{\mu} = (\mathbf{x}, t, s),$$

as well as the following 5-momentum:

$$p_{\mu} \equiv -\mathrm{i}\partial_{\mu} = (-\mathrm{i}\nabla, -\mathrm{i}\partial_t, -\mathrm{i}\partial_s),$$

so that, with the usual identification $E = \mathrm{i}\partial_t$, and with $m = \mathrm{i}\partial_s$, we obtain

$$p_{\mu} = (\mathbf{p}, -E, -m), \qquad p^{\mu} = g^{\mu\nu} p_{\nu} = (\mathbf{p}, m, E).$$

Thereupon the mass does not enter as an external parameter, but as a remnant of the fifth component of the particle's momentum. Hereafter, the 5-momentum operator will act on a massless field so that

$$\partial_5 A = \partial_s A = 0.$$

Now let us set up the five-dimensional quantities that allow us to retrieve the two Galilean limits of electromagnetism. They are given by defining two embeddings of the 5-potential:

$$A_{\mu} = (\mathbf{A}, A_4, A_5).$$

Under the transformation in equation (35) its components transform, from equation (37), as

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \mathbf{v}A_5, \\ A_{4'} &= A_4 + \mathbf{v}\cdot\mathbf{A} + \tfrac{1}{2}\mathbf{v}^2 A_5, \\ A_{5'} &= A_5. \end{aligned} \tag{38}$$

Next, we write the five-dimensional electromagnetic antisymmetric Faraday tensor:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & b_3 & -b_2 & c_1 & d_1 \\ -b_3 & 0 & b_1 & c_2 & d_2 \\ b_2 & -b_1 & 0 & c_3 & d_3 \\ -c_1 & -c_2 & -c_3 & 0 & a \\ -d_1 & -d_2 & -d_3 & -a & 0 \end{pmatrix}. \tag{39}$$

Thus, we have

$$\begin{aligned} \mathbf{b} &= \nabla \times \mathbf{A}, \\ \mathbf{c} &= \nabla A_4 - \partial_4 \mathbf{A}, \\ \mathbf{d} &= \nabla A_5 - \partial_5 \mathbf{A}, \\ a &= \partial_4 A_5 - \partial_5 A_4. \end{aligned} \tag{40}$$

The 5-current,

$$j_\mu = (\mathbf{j}, j_4, j_5),$$

transforms under the transformation, equation (35), as

$$\begin{aligned} \mathbf{j}' &= \mathbf{j} + \mathbf{v} j_5, \\ j_{4'} &= j_4 + \mathbf{v}\cdot\mathbf{j} + \tfrac{1}{2}\mathbf{v}^2 j_5, \\ j_{5'} &= j_5. \end{aligned} \tag{41}$$

The continuity equation takes the form

$$\partial^\mu j_\mu = \nabla\cdot\mathbf{j} - \partial_4 j_5 - \partial_5 j_4 = 0. \tag{42}$$

The five-dimensional Lorenz-like condition takes a similar form:

$$\partial^\mu A_\mu = \nabla\cdot\mathbf{A} - \partial_4 A_5 - \partial_5 A_4 = 0. \tag{43}$$

In the presence of sources, the Maxwell equations are

$$\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0, \tag{44}$$

and

$$\partial_\nu F^{\mu\nu} = j^\mu, \tag{45}$$

so that in terms of the components of F defined in equation (39), we find, from equation (44)

$$\begin{aligned} &\nabla\cdot\mathbf{b} = 0, \\ &\nabla\times\mathbf{c} + \partial_4\mathbf{b} = \mathbf{0}, \\ &\nabla\times\mathbf{d} + \partial_5\mathbf{b} = \mathbf{0}, \\ &\nabla a - \partial_4\mathbf{d} + \partial_5\mathbf{c} = \mathbf{0}, \end{aligned} \tag{46}$$

whereas equation (45) reduces to

$$\begin{aligned} &\nabla\times\mathbf{b} - \partial_5\mathbf{c} - \partial_4\mathbf{d} = \mathbf{j}, \\ &\nabla\cdot\mathbf{c} - \partial_4 a = -j_4, \\ &\nabla\cdot\mathbf{d} + \partial_5 a = -j_5. \end{aligned} \tag{47}$$

From $F_{\mu'\nu'} = \Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}F_{\alpha\beta}$, the entries of F in equation (37) transform as

$$\begin{aligned} a' &= a + \mathbf{v}\cdot\mathbf{d}, \\ \mathbf{b}' &= \mathbf{b} - \mathbf{v}\times\mathbf{d}, \\ \mathbf{c}' &= \mathbf{c} + \mathbf{v}\times\mathbf{b} + \tfrac{1}{2}\mathbf{v}^2\mathbf{d} - a\mathbf{v} - \mathbf{v}(\mathbf{v}\cdot\mathbf{d}), \\ \mathbf{d}' &= \mathbf{d}. \end{aligned} \tag{48}$$

Let us now see how the electric and magnetic limits are contained within the previous formulae.

3.2.1. Electric limit

As mentioned previously, the electric limit is characterized by 4-potential and 4-current vectors which are timelike, that is, their time component is much larger than the length of their spatial components. In the reduction approach, it corresponds to defining the embedding of the potentials and currents as

$$(\mathbf{A}_e, V_e) \hookrightarrow A_e = (\mathbf{A}_e, 0, -\mu_0\epsilon_0 V_e), \tag{49}$$

and

$$(\mathbf{j}_e, \rho_e) \hookrightarrow j_e = (\mu_0\mathbf{j}_e, 0, -\mu_0\rho_e), \tag{50}$$

respectively.

From equations (38) and (49) we retrieve equation (12). Similarly we obtain equation (16) from equations (41) and (50). As for the continuity equation, equation (42), it becomes equation (17). From the first line of equation (40), we come to the natural definition:

$$\mathbf{B}_e \equiv \mathbf{b} = \nabla\times\mathbf{A}_e.$$

The electric field is defined as the component $\mathbf{d}$, so that from the third line of equation (40) we have $\mathbf{E}_e \equiv \frac{1}{\mu_0\epsilon_0}\mathbf{d} = -\nabla V_e$, as in equation (13). From equation (40), we note that $\mathbf{c} = -\partial_t\mathbf{A}_e$ and $a = -\frac{1}{\mu_0\epsilon_0}\partial_t V_e$. Then, equation (48) leads to equation (8). The corresponding Maxwell equations, equation (19), are obtained from equations (46) and (47).

Note that the second line of equation (47) provides a condition similar to Lorenz gauge fixing:

$$\nabla\cdot\mathbf{A}_e = -\mu_0\epsilon_0\partial_t V_e.$$

This expression may also be obtained by substituting equation (49) into equation (43).

3.2.2. Magnetic limit

This non-relativistic limit is characterized by spacelike 4-potential and 4-current vectors; their time component is small compared to the length of their spatial components. Hereafter, we show that it corresponds to defining the embedding of the potentials and currents as

$$(\mathbf{A}_m, V_m) \hookrightarrow A_m = (\mathbf{A}_m, -V_m, 0), \tag{51}$$

and

$$(\mathbf{j}_m, \rho_m) \hookrightarrow j_m = \left(\mu_0\mathbf{j}_m, -\frac{1}{\epsilon_0}\rho_m, 0\right) \tag{52}$$

respectively.

From equations (38) and (51), we retrieve equation (10). Similarly equations (41) and (52) lead to equation (14), and the continuity equation (42) gives equation (15), which shows that the current $\mathbf{j}_m$ cannot be related to a convective transport of charge! As above, we define the magnetic field as $\mathbf{B}_m \equiv \mathbf{b} = \nabla \times \mathbf{A}_m$ and the electric field is now defined as the component $\mathbf{c}$, so that from the second line of equation (40) we obtain $\mathbf{E}_m \equiv \mathbf{c} = -\nabla V_m - \partial_t \mathbf{A}_m$, as in equation (11). Then equation (48) leads to equation (7). The Maxwell equations (20) are obtained from equations (46) and (47). Finally, note that by replacing equation (51) in equation (43), we obtain Coulomb's gauge condition

$$\nabla \cdot \mathbf{A}_m = 0.$$

4. Some applications

Clearly, Galilean theories are limits of Maxwell's theory. Our claim is not that Galilean theories represent an alternative to special relativity in the low velocity regime, but that one must be vigilant when taking low velocity limits in order to make sure that the ensuing equations satisfy appropriate invariant properties. This is of the utmost importance nowadays because of recent dramatic breakthroughs and intense research in condensed matter physics. A consequence of Le Bellac and Lévy-Leblond's work is that more physical systems could have been described properly with the correct formulation of Galilean electromagnetism.

In a future work, we plan to collect some results about phenomena that could not supposedly be explained by Galilean theories because the two (magnetic and electric) limits were incoherently mixed. In this section, we provide some examples borrowed from fluid dynamics and the physics of continuous media.

Within the context of Galilean covariance, magnetohydrodynamics (MHD) and electrohydrodynamics (EHD) turn out to be two different sets of approximations of the Maxwell equations where retardation (and, therefore, wave characteristics) has been neglected. Indeed, effects that are important in MHD become marginal in EHD (such as the curl of the electric field, as in equation (19)) and vice versa (e.g., the displacement current is negligible in MHD (as in equation (20)). Melcher has greatly clarified these facts by disjoining the electroquasistationary approximation used in EHD from the magnetoquasistationary approximation of MHD [20]. From section 3.2 of Melcher and Haus [20], we see that the underlying equations are precisely the same as the Galilean limits, equations (19) and (20). Melcher's main argument relies on the comparison of three characteristic time scales (magnetic diffusion time, charge relaxation time and wave transit time). The third one is the square root of the product of the two former time scales. As an example, in the magnetic limit, the charge relaxation time scale is very small and the magnetic field does have enough time to diffuse inside the Ohmic carrier. It is straightforward to see that the electroquasistatic of Melcher corresponds to the electric limit whereas the magnetic limit is just the magnetoquasistatic. Hence, a large amount of our technology is based on Galilean electromagnetism as soon as waves are neglected.

Now, let us comment on the low-velocity electrodynamics of continuous media. We find in some textbooks that a dielectric in motion is characterized by the presence of a motional polarization given by

$$\mathbf{P}' = \epsilon_0 \chi (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{53}$$

where χ is the dielectric susceptibility [21]. Lorentz utilized it in order to derive the Fresnel–Fizeau formula at first order (see [22] and pp 174–6 of [21]). This formula can be misleading.

Maxwell equations in continuous media are covariant under the Poincaré–Lorentz transformations [23]:

$$\nabla \cdot \mathbf{B} = 0,$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \partial_t \mathbf{D}.$$

In continuous media, the constitutive equations relate the excitation **D**, the field **E** and the polarization **M**:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}.$$

These relations are valid in both Galilean and Einsteinian relativity [23]. Let us examine the electromagnetic laws when we take into account the motion of a medium at low velocity [20].

First, we recall that the Galilean transformation, equation (6), leads to equation (9) for the differentiation operators. From this equation, together with the identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}),$$

we find

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}'}{\partial t} + \mathbf{v}(\nabla \cdot \mathbf{A}') - \nabla \times (\mathbf{v} \times \mathbf{A}'),$$

for any vector **A**. If we utilize these transformations with the two Galilean limits expressed for a continuous medium, the Maxwell equations in a reference frame moving at velocity **v** read from equation (9), these equations transform in such a way that we can deduce the following field transformations.

The formula, equation (53), used by Lorentz is not compatible with Galilean relativity. However, the electric field and the magnetic field which create the polarization in Fizeau experiment come from a light wave, so that Lorentz was right to use this formula in order to derive a first-order relativistic effect, even though there is no contradiction with the electric limit formula $\mathbf{P}' = \mathbf{P} = \epsilon_0 \chi \mathbf{E}' = \epsilon_0 \chi \mathbf{E}$.

Magnetic limit	*Electric limit*
$\nabla' \times \mathbf{H}' = \mathbf{j}'$,	$\nabla' \times \mathbf{E}' = \mathbf{0}$,
$\nabla' \cdot \mathbf{B}' = 0$,	$\nabla' \cdot \mathbf{D}' = \rho'$,
$\nabla' \cdot \mathbf{j}' = 0$,	$\nabla' \cdot \mathbf{j}' + \dfrac{\partial \rho'}{\partial t'} = 0$,
$\partial_{t'} \mathbf{B}' = -\nabla' \times \mathbf{E}'$,	$\nabla' \times \mathbf{H}' = \mathbf{j}' + \dfrac{\partial \mathbf{D}'}{\partial t'}$.
$\mathbf{B} = \mathbf{B}'$,	$\mathbf{E} = \mathbf{E}'$,
	$\rho = \rho'$,
$\mathbf{j} = \mathbf{j}'$,	$\mathbf{j} = \mathbf{j}' + \rho' \mathbf{v}$,
$\mathbf{H} = \mathbf{H}'$,	$\mathbf{H} = \mathbf{H}' + \mathbf{v} \times \mathbf{D}'$,
$\mathbf{E} = \mathbf{E}' - \mathbf{v} \times \mathbf{B}'$,	$\mathbf{D} = \mathbf{D}'$,
$\mathbf{M} = \mathbf{M}'$,	$\mathbf{P} = \mathbf{P}'$,
$\mathbf{P} = \mathbf{P}' + \mathbf{v} \times \mathbf{M}'/c^2$,	$\mathbf{M} = \mathbf{M}' - \mathbf{v} \times \mathbf{P}'$.

We plan to return to these questions in more detail soon. We will also discuss other applications, such as in superconductivity, the Schrödinger equation and Lévy-Leblond equation with external fields, and the Trouton–Noble experiment, among others.

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