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Amplitude equations for mechanical analogues of Faraday and nonlinear optical rotations

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Abstract
What is the relationship between the propagation of a light wave in a Kerr medium in the presence of a magnetic field and the oscillation of a spherical pendulum on a rotating platform? We apply the general formalism of amplitude equations in order to explore both the nonlinear and the rotation-induced precession of these optical and mechanical experiments. Then, we explain the surprising analogies between the two physical phenomena.

1. Introduction

The conical pendulum was used first by Robert Hooke to approach the notion of central force [1, 2]: when the pendulum made up of a heavy mass representing the Earth hanging from a wire is moved away from its equilibrium position vertically from the point of suspension, it undergoes a restoring force which tends to bring it back to the centre similarly to the gravitational force exerted by the Sun on the Earth. For small amplitudes, the trajectories are ellipses which precess. The ellipses are centred on the axis in contrast to the case of planets where the attractive centre corresponds to one of the foci of the elliptical path.

In this work, we will revisit the classical works [3–8] on the precession associated with a conical pendulum from the modern viewpoint of an amplitude equations formalism. In particular, we will show that some optical phenomena such as nonlinear and Faraday rotations can be understood very easily thanks to mechanical analogues related to the conical pendulum's behaviour.

1.1. Mechanical analogue for nonlinear optical rotation

Several years ago, Maker and Terhune [9] demonstrated the occurrence of optical effects due to an induced polarization third order in the electric field strength [10]. In particular, they explained quantum mechanically the precession of an elliptic light ray due to the nonlinearity of the refractive index (cf appendix A). Indeed, a light ray with an elliptic polarization is compelled to precess around the direction of propagation and the magnitude of the effect depends on the square of the electric field and the light pulsation. As a matter of fact, the ellipse can be decomposed into two counter-rotating spherical polarizations with different amplitudes and the nonlinearity modifies the optical indices associated with both polarizations which results in a precession phenomenon (cf appendix A). For the influence of nonlinearity in optics, the reader is referred to the book of Newell and Moloney [11].

We will show that this optical precession has a mechanical counterpart. Indeed, G B Airy [3, 4] demonstrated in 1851 that a conical pendulum whose initial motion was elliptical was compelled to precess in the same direction as the oscillation of its mass [5–8, 12–15]. If the length of (the conical) pendulum is l, the semi-major axis of the ellipse described by the pendulum bob is a, and the semi-minor axis is b, then the line of the apses of the ellipse will perform a complete revolution in the time of a complete double vibration (i.e. the time of describing the ellipse) multiplied by $\frac{8}{3}\frac{l^2}{ab}$.

The phenomenon was well known to L Foucault who avoided the elliptical motion because it would pollute experiments designed to reveal the Earth's rotation which also induced an elliptical precession. Indeed, Foucault used a special set-up in order to launch his pendulum initially with a linear polarization whose plane of oscillation would only precess because of the Earth's rotation and not because of the Airy precession [14].

1.2. Mechanical analogue for Faraday optical rotation

Another optical rotation can be matched with a mechanical analogue in the framework of an amplitude equations formalism. In 1845, M Faraday discovered that a linearly polarized light ray was induced to rotate by an angle θ when travelling through a medium of length L submitted to a static magnetic field B in the direction of the wavevector [10, 16] $\theta = VBL$ where V is the Verdet constant which depends on the medium's properties (cf appendix B). If a mirror is placed at the end of the medium and reflects the ray, the total rotation after the round trip is twice θ and is not null as one would expect for classical optical activity in a chiral medium, which demonstrates that the magnetic field is an axial vector.

The Earth's rotation is also an axial vector and one would expect similar behaviour of the plane of oscillation for the pendulum. We will show that this is indeed the case with the help of amplitude equations for both phenomena.

2. Amplitude equations for mechanical analogues of Faraday and nonlinear optical rotation

2.1. Amplitude equations for the linear Foucault pendulum

Using Newtonian or Lagrangian mechanics [17], it is straightforward to show that the horizontal coordinates (x, y) of the pendulum satisfies the following equations:

$$\ddot{x} - 2\Omega_T \sin\lambda\, \dot{y} + \omega_0^2 x = 0 \tag{1}$$

$$\ddot{y} + 2\Omega_T \sin\lambda\, \dot{x} + \omega_0^2 y = 0 \tag{2}$$

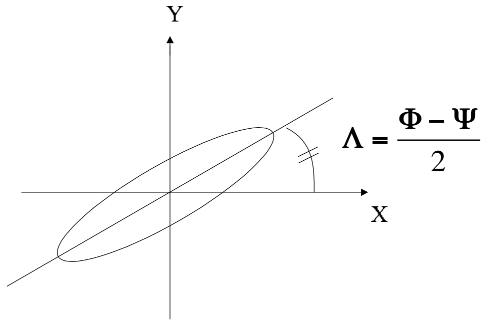

Figure 1. Angle of precession of the elliptic path ($\Lambda = (\Phi - \Psi)/2$).

where Ω_T is the Earth's rotation, λ the latitude and $\omega_0 = \sqrt{g/l}$ is the natural pulsation of the pendulum expressed as a function of the gravity g and its length l and where we used the approximations $x, y \ll l$.

We define the complex position of the pendulum $Z = x + \mathrm{i}y$ in order to write the equation for the motion in the horizontal plane:

$$\ddot{Z} + 2\mathrm{i}\Omega_T \sin\lambda \dot{Z} + \omega_0^2 Z = 0 \tag{3}$$

whose solution is

$$Z = [A_0 \exp(\mathrm{i}\omega_0 t) + B_0 \exp(-\mathrm{i}\omega_0 t)] \exp(-\mathrm{i}\Omega_T \sin\lambda t) \tag{4}$$

where we used $\Omega_T \ll \omega_0$ and where $A_0 = R\exp(\mathrm{i}\Phi)$ and $B_0 = S\exp(-\mathrm{i}\Psi)$ are complex constants depending on the initial conditions (the way the pendulum was launched ...).

When $A_0 = \pm B_0$, the motion passes through the equilibrium point. If $\Omega_T = 0$, the motion resumes to that of a simple pendulum with period $T_0 = 2\pi/\omega_0$ with an elliptical trajectory given by

$$Z_e = A + B = A_0 \exp(\mathrm{i}\omega_0 t) + B_0 \exp(-\mathrm{i}\omega_0 t). \tag{5}$$

The corresponding amplitude equations for the two circular polarizations without the Earth's rotation are $\dot{A} = \mathrm{i}\omega_0 A$ and $\dot{B} = -\mathrm{i}\omega_0 B$. We use the terminology 'circular polarization' in the mechanical context in order to describe the left and right motion of the pendulum whose different amplitudes are recomposed to describe the elliptical motion.

The axis of this ellipse does not change with time. The same situation is encountered at the equator (with $\lambda = 0$).

The trajectory can be written in a dimensionless form (formally $\omega_0 = 1$),

$$Z_e^* = R\exp[\mathrm{i}(t + \Phi)] + S\exp[-\mathrm{i}(t + \Psi)] \tag{6}$$

with $\ddot{Z}_e^* + Z_e^* = 0$ for the linear conical pendulum.

One can remark that the origin of time is arbitrary. If we define $\Lambda = (\Phi - \Psi)/2$ such that

$$Z_e^* = \exp(\mathrm{i}(\Phi - \Psi)/2)[R\exp[\mathrm{i}(t + (\Phi + \Psi)/2)] + S\exp[-\mathrm{i}(t + (\Phi + \Psi)/2)]], \tag{7}$$

then we get by redefining the origin of time ($t_0 = -(\Phi + \Psi)/2$),

$$Z_e^* = \exp(\mathrm{i}\Lambda)[R\exp(\mathrm{i}\tau) + S\exp(-\mathrm{i}\tau)] \tag{8}$$

with $\tau = t - t_0$.

By changing to the frame of reference turning with angle Λ, the elliptic path $z = Z_e^* \exp(-\mathrm{i}\Lambda)$ becomes (figure 1)

$$z = [R\exp(\mathrm{i}\tau) + S\exp(-\mathrm{i}\tau)] \tag{9}$$

with its complex conjugate,

$$\bar{z} = [S\exp(\mathrm{i}\tau) + R\exp(-\mathrm{i}\tau)]. \tag{10}$$

Using Cramer's determinants formula, one solves the last system of equations for z and $\bar{z}$:

$$\exp(\mathrm{i}\tau) = \frac{Rz - S\bar{z}}{R^2 - S^2} \tag{11}$$

and

$$\exp(-\mathrm{i}\tau) = \frac{R\bar{z} - Sz}{R^2 - S^2}. \tag{12}$$

If one multiplies the above equations, one deduces the equation for the elliptic trajectory,

$$\frac{X^2}{(R+S)^2} + \frac{Y^2}{(R-S)^2} = 1 \tag{13}$$

using $z = X + \mathrm{i}Y$, $X = x/l$ and $Y = y/l$ the dimensionless horizontal coordinates. We finally get the dimensionless values for the major and minor semi-axis of the ellipse:

$$a^* = a/l = (R+S) \qquad \text{and} \qquad b^* = b/l = (R-S). \tag{14}$$

If $\Omega_T \neq 0$ and if $\lambda \neq 0$, the complex vector Z_e rotates (in the frame of the Earth) with angular frequency $-\Omega_T \sin\lambda$, i.e. with a period $2\pi/(\Omega_T \sin\lambda) = T_{\mathrm{earth}}/(\sin\lambda)$ where T_{earth} is the rotation period of the Earth.

The corresponding amplitude equations for the two 'circular polarizations' with the Earth's rotation are $\dot{A} = \mathrm{i}(\omega_0 - \Omega_T \sin\lambda)A$ and $\dot{B} = -\mathrm{i}(\omega_0 + \Omega_T \sin\lambda)B$. The Foucault precession can be interpreted as a kind of Doppler effect for the two circular polarizations as the frequencies of oscillation are modified by the Earth's rotation in the same way (i.e. with a negative sign): the total frequency of the B polarization is increased compared to that of the A polarization which is diminished.

2.2. Amplitude equations for the nonlinear conical pendulum

We solve now the equation for the conical pendulum without rotation taking into account its intrinsic nonlinearity. The dot corresponds to the partial derivative with respect to the dimensionless time $t^* = t\omega_0$. From Lagrangian mechanics (cf appendix C), one deduces the nonlinear equation for the complex position with the dimensionless variables:

$$\ddot{Z} + Z - \tfrac{1}{2}|Z|^2 Z + |\dot{Z}|^2 Z = 0 \tag{15}$$

which we solve with

$$Z = A\exp(\mathrm{i}t) + B\exp(-\mathrm{i}t). \tag{16}$$

For slow changes of the amplitudes with time ($\dot{A} \ll 1$ and $\dot{B} \ll 1$), we use the approximation

$$\dot{Z} \simeq \mathrm{i}(A\exp(\mathrm{i}t) - B\exp(-\mathrm{i}t)) \tag{17}$$

in order to get

$$\ddot{Z} + Z \simeq 2\mathrm{i}(\dot{A}\exp(\mathrm{i}t) - \dot{B}\exp(-\mathrm{i}t)) \tag{18}$$

where in addition, the slowly varying envelope approximation was introduced: $\ddot{A} \ll \dot{A}$ and $\ddot{B} \ll \dot{B}$.

This leads to a set of coupled amplitude equations for both A and B:

$$\dot{A} = \mathrm{i}A(\alpha|A|^2 + \beta|B|^2) \tag{19}$$

and

$$\dot{B} = -\mathrm{i}B(\alpha|B|^2 + \beta|A|^2) \tag{20}$$

with $\alpha = 1/4$ and $\beta = -1/2$.

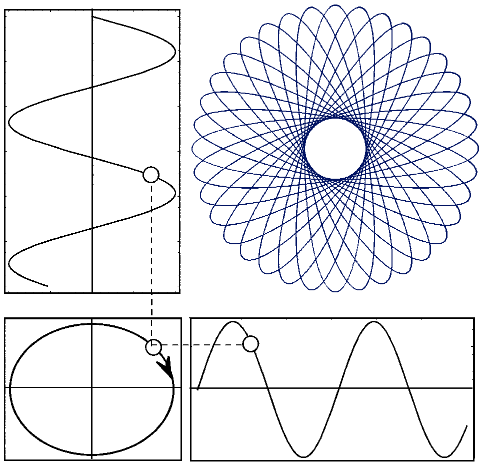

Figure 2. Decomposition of the rosette-like trajectory into two separate oscillations with different amplitudes.

One obtains a rosette-like trajectory which can be interpreted as the sum of two oscillations with different amplitude (figure 2).

Moreover, one notes that for a conical nonlinear pendulum,

$$\dot{A} = \mathrm{i}A\left(\tfrac{1}{4}|A|^2 - \tfrac{1}{2}|B|^2\right), \qquad (21)$$

the sign in front of the nonlinear term in A is opposite to the the usual case of a nonlinear 'linear' (i.e. not conical) pendulum:

$$\dot{A} = -\frac{\mathrm{i}}{4}|A|^2 A. \qquad (22)$$

However, if one puts $A = B$ in the amplitude equation for the nonlinear conical pendulum, then one recovers the latter equation for the 'linear' pendulum (cf appendix D).

Now, one infers that the moduli of the amplitudes are constant ($|A| = A_0 = R$, $|B| = B_0 = S$) because

$$\partial_{t^*}|A|^2 = 0 \qquad \text{and} \qquad \partial_{t^*}|B|^2 = 0. \qquad (23)$$

The amplitude equations are simpler now,

$$\dot{A} = \mathrm{i}A(\alpha|A_0|^2 + \beta|B_0|^2) \qquad (24)$$

and

$$\dot{B} = -\mathrm{i}B(\alpha|B_0|^2 + \beta|A_0|^2). \qquad (25)$$

One notes that these equations are invariant by rotation,

$$A \to A\exp\mathrm{i}\varphi \qquad \text{and} \qquad B \to B\exp\mathrm{i}\varphi \qquad (26)$$

and they can be solved with $A = R\exp(\mathrm{i}\Phi) = R\exp(\mathrm{i}\omega_A t^*)$ and $B = S\exp(-\mathrm{i}\Psi) = S\exp(-\mathrm{i}\omega_B t^*)$ which leads to $\omega_A = \alpha R^2 + \beta S^2$ and $\omega_B = \alpha S^2 + \beta R^2$.

Then, one infers the angle of precession of the elliptic path (figure 3):

$$\Lambda = \frac{\Phi - \Psi}{2} = \frac{\omega_A - \omega_B}{2} t^* \qquad (27)$$

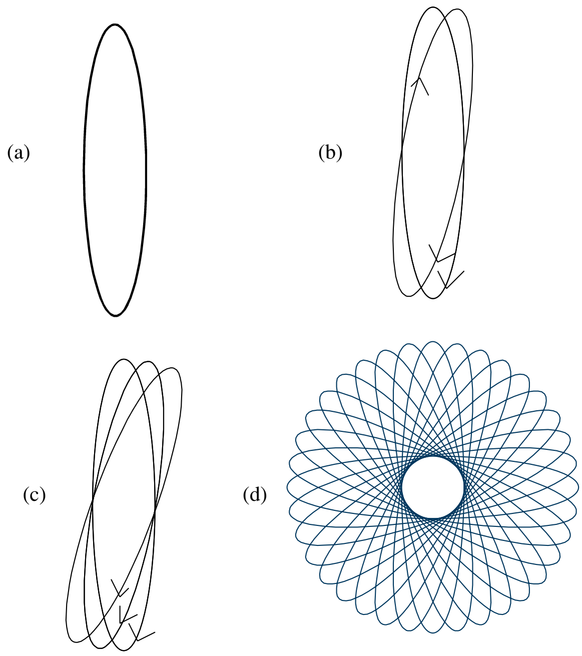

Figure 3. Trajectory of the pendulum projected on the horizontal plane: (a) ellipse after one period, (b) precessing ellipses after two periods, (c) precessing ellipses after three periods and (d) precessing ellipses after *N* periods.

that is,

$$\Lambda = \frac{\alpha - \beta}{2} a^* b^* t^* \tag{28}$$

with $a^* = R + S$ ($b^* = R - S$), the major (minor) semi-axis of the elliptic path. One concludes that the angular velocity of precession is proportional to the surface of the ellipse (πab):

$$\Omega_p^* = \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \frac{\alpha - \beta}{2} a^* b^* = \frac{3}{8} a^* b^* = \frac{3}{8}(R^2 - S^2) \tag{29}$$

as was discovered by Airy with the dimensional form,

$$\Omega_p = \frac{3}{8}(R^2 - S^2)\omega_0 = \frac{3}{8}\frac{ab}{l^2}\sqrt{\frac{g}{l}}. \tag{30}$$

Now, if one is interested in the total angle Λ_T performed during one period of oscillation $T = 2\pi/\omega_0$, one finds

$$\Lambda_T = \frac{3}{8}\frac{ab}{l^2}\sqrt{\frac{g}{l}}T = \frac{3}{4}\frac{\pi ab}{l^2} = \frac{3}{4}\Omega_S \tag{31}$$

where $\Omega_S = \iint_S \mathrm{d}S/r^2 = \pi ab/l^2$ is the solid angle described by the elliptic path and seen from the fixed point of attachment.

As noted by J Larmor commenting on J Clerk Maxwell's matter and motion [18], if the mass were free to rotate on the wire as axis (free point of attachment), it would turn through Ω_S (and not $3/4\Omega_S$) during each revolution.

2.3. Amplitude equations for the nonlinear Foucault pendulum

Now, we would like to solve the equation for the conical pendulum with both intrinsic nonlinearity and external rotation:

$$\ddot{Z} + Z - \tfrac{1}{2}|Z|^2 Z + |\dot{Z}|^2 Z = 0 \tag{32}$$

with the following ansatz: $Z = W \exp(\mathrm{i}\Omega t)$ which consists in changing from the rotative frame of reference to the stationary one. The nonlinear Foucault equation for W is

$$\ddot{W} + 2\mathrm{i}\Omega\dot{W} - \Omega^2 W + W - \tfrac{1}{2}|W|^2 W + |\dot{W}|^2 W - \mathrm{i}\Omega\dot{W}|W|^2 + \mathrm{i}\Omega W^2\overline{\dot{W}} + \Omega^2|W|^2 W = 0 \tag{33}$$

where we recognize a Coriolis-like term ($2\mathrm{i}\Omega\dot{W}$) and a centrifugal-like term ($-\Omega^2 W$).

If one neglects the nonlinear terms in W and Ω, one recovers the Foucault equation for W:

$$\ddot{W} + 2\mathrm{i}\Omega\dot{W} + W = 0. \tag{34}$$

Now, one solves the nonlinear equation for W by introducing the following solution mimicking the linear resolution:

$$W = A\exp(\mathrm{i}t) + B\exp(-\mathrm{i}t) \tag{35}$$

with (as usual now)

$$\dot{W} \simeq \mathrm{i}(A\exp(\mathrm{i}t) - B\exp(-\mathrm{i}t)) \tag{36}$$

where we used the slowly varying envelope approximation as before without rotation.

One ends up with the equation for the A polarization:

$$\dot{A} = -\frac{\mathrm{i}A}{2}[\delta_0 + \delta_1|A|^2 + \delta_2|B|^2] \tag{37}$$

with $\delta_0 = 2\Omega + \Omega^2$, $\delta_1 = -(1/2 + 2\Omega + \Omega^2)$ and $\delta_2 = 1 + 2\Omega - 2\Omega^2$.

If $\Omega = 0$, one has $\delta_0 = 0$, $\delta_1 = -1/2$ and $\delta_2 = 1$. One recovers the amplitude equation for the A polarization of the simple nonlinear conical pendulum.

If $\Omega \ll 1$ (i.e. $\Omega \ll \Omega^2$), one obtains

$$\dot{A} = -\frac{\mathrm{i}A}{2}\left[+2\Omega - \frac{1}{2}|A|^2 + |B|^2\right]. \tag{38}$$

The equation for the B polarization becomes

$$\dot{B} = -\frac{\mathrm{i}B}{2}[\gamma_0 + \gamma_1|A|^2 + \gamma_2|B|^2] \tag{39}$$

with $\gamma_0 = 2\Omega - \Omega^2$, $\gamma_1 = 1/2 - 2\Omega + \Omega^2$ and $\gamma_2 = -1 + 2\Omega + 2\Omega^2$.

If $\Omega = 0$, one has $\gamma_0 = 0$, $\gamma_1 = 1/2$ and $\gamma_2 = -1$. One recovers the amplitude equation for the B polarization of the simple nonlinear conical pendulum.

If $\Omega \ll 1$ (i.e. $\Omega \ll \Omega^2$), one obtains

$$\dot{B} = -\frac{\mathrm{i}B}{2}\left[+2\Omega + \frac{1}{2}|B|^2 - |A|^2\right]. \tag{40}$$

One notes that for the two circular polarizations, the axial feature of the rotation vector leads to the same sign for the terms proportional to Ω whereas the nonlinear terms are of opposite sign in the amplitude equations for A and B. Hence, this illustrates once again the fact that the Coriolis effect in mechanics is similar to the Faraday rotation in optics.

We have looked for instabilities associated with an angular parametric excitation of the pendulum. We note that the pendulum should be driven at twice its natural frequency.

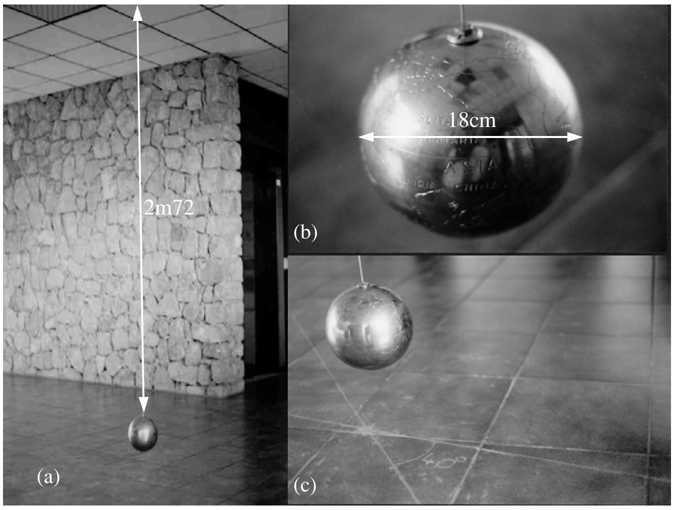

Figure 4. Experimental set-up: (a) the spherical pendulum, (b) the oscillating mass and (c) the method of measurement.

The parametric driving for a conical pendulum is such that the A (B) polarization is modified by a pumping term proportional to the B (A) polarization: it is very different from the usual pendulum with vertical excitation of the point of fixation where the parametric term is proportional to the same polarization. We have shown that the angular parametric driving does not induce an instability of the conical pendulum (contrary to the usual pendulum with vertical driving of its point of fixation) but only a modulation of the amplitudes of its two polarizations.

3. Experiments

In order to verify Airy's precession law, we built a conical pendulum with a hollow globe of brass of diameter 18 cm which we filled with lead beads: the mass is of the order of 20 kg. We suspend it with a steel cable of length 2.72 m. Hence, the small angle approximation is fulfilled with such a length compared to the horizontal projection of the trajectory. In addition, air motion will not perturb such a weight too much (figures 4(a) and (b)).

For measurements, one draws two axes on the soil with a predefined angle (say 40). Then, one launches the pendulum. When its path of oscillation crosses the first axis parallel to it, we start the chronometer and we stop it when the pendulum reaches the second axis. In the same time, one marks the lengths of the major axis with a chalk and the minor axis with a vertical plane which allows us to draw on the soil the projection of the tangents to the trajectory perpendicularly to both axes of the ellipse. Hence, one has the area and the time during the precession between the two axes of reference (figure 4(c)).

The comparison between theory and experiments is in good agreement with respect to the ratio between the angular precession and the elliptic area (figure 5): 2.5×10^{-2} for the experiments and 2.6×10^{-2} for the theory. This coincidence is quite successful despite the numerous effects which would lead to disagreement [19]: the elasticity/torsion of the cable, air drag, the role of the fixation and second-order effects.

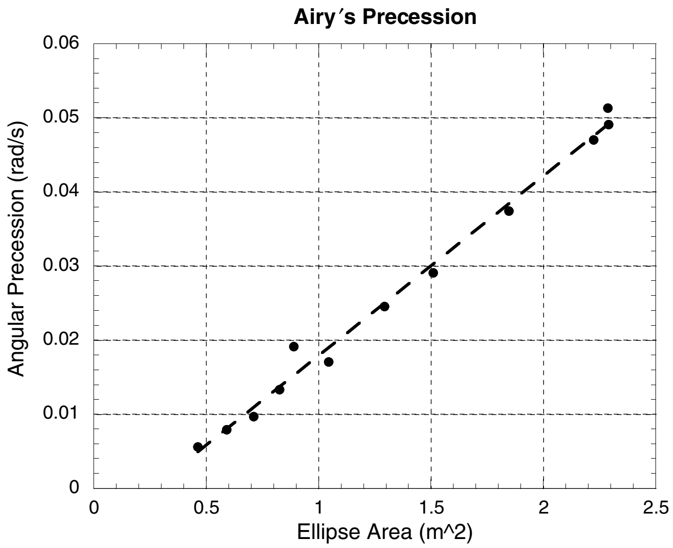

Figure 5. Experimental results: angle of precession of the elliptic path as a function of the ellipse area.

We were not able to find in the literature the optical counterpart of our experiments. Indeed, one usually displays the precession rate as a function of the frequency for a given ellipticity. The reader will find this type of measurement in [20–22].

4. Conclusions

The amplitude equation formalism is a powerful technique for displaying the universal behaviour of apparently disconnected physical phenomena. By revisiting the seminal works on the conical pendulum, we were able to show its close relationship to the behaviour of the polarization of light waves inside nonlinear media in the presence of a magnetic field. We hope that our study will clarify the physical pictures associated with such optical processes. We have shown that the induced precession due to the intrinsic nonlinearity is proportional to the area swept out by the elliptical path of either the electric field or the pendulum's mass. In addition, the precession is linear with the natural frequency of the phenomena either for the nonlinearities or the rotation (mechanical or magnetic). Concerning the rotation, this result is well known when one deals with the so-called Larmor precession in classical electromagnetism as the effect of a magnetic field can be interpreted as an angular rotation. Our analysis does not exhaust the scope of finding mechanical analogues of nonlinear optics and this field deserves much more exploration. As stated by Lord Kelvin a long time ago: It seems to me that the test of 'Do we or do we not understand a particular point in physics?' is, 'Can we make a mechanical model of it?'

Acknowledgments

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Appendix A

The constitutive equation of an isotropic material for a polarization which is third order in the electric field strength is [9]

$$\mathbf{P} = \epsilon_0[6\chi_{1122}(\mathbf{E}.\mathbf{E}^*)\mathbf{E} + 3\chi_{1221}(\mathbf{E}.\mathbf{E})\mathbf{E}^*] \tag{A.1}$$

where the numerical subscripts 1, 2 and 3 stand for x, y and z. One decomposes the electric field in right and left polarizations:

$$\mathbf{E} = E_+\hat{\sigma}_+ + E_-\hat{\sigma}_- \tag{A.2}$$

with

$$\hat{\sigma}_\pm = \frac{\hat{x} \pm \mathrm{i}\hat{y}}{\sqrt{2}} \tag{A.3}$$

which obeys the following relations:

$$\hat{\sigma}_\pm^* = \hat{\sigma}_\mp, \qquad \hat{\sigma}_\pm \cdot \hat{\sigma}_\pm = 0, \qquad \hat{\sigma}_\pm \cdot \hat{\sigma}_\mp = 1. \tag{A.4}$$

One deduces

$$\mathbf{E}.\mathbf{E}^* = |E_+|^2 + |E_-|^2 \qquad \text{and} \qquad \mathbf{E}.\mathbf{E} = 2E_+E_-. \tag{A.5}$$

Hence, we have

$$\mathbf{P} = \epsilon_0[6\chi_{1122}(|E_+|^2 + |E_-|^2)\mathbf{E} + 6\chi_{1221}(E_+E_-)\mathbf{E}^*] \tag{A.6}$$

which we can rewrite in the form

$$\mathbf{P} = P_+\hat{\sigma}_+ + P_-\hat{\sigma}_- \tag{A.7}$$

and which leads to

$$P_\pm = \epsilon_0[6\chi_{1122}|E_\pm|^2E_\pm + 6(\chi_{1122} + \chi_{1221})|E_\mp|^2E_\pm] \tag{A.8}$$

that is

$$P_\pm = \epsilon_0\chi_\pm^{NL}E_\pm \tag{A.9}$$

with

$$\chi_\pm^{NL} = 6\chi_{1122}|E_\pm|^2 + 6(\chi_{1122} + \chi_{1221})|E_\mp|^2. \tag{A.10}$$

The electric field is the solution of the Maxwell equations,

$$\nabla^2 E_\pm(\omega, \mathbf{r}) = \mu_0\epsilon_\pm\frac{\partial^2 E_\pm(\omega, \mathbf{r})}{\partial t^2}, \tag{A.11}$$

where we have introduced the dielectric constants and the dielectric index for both the vacuum (subscript 0) and the medium:

$$\epsilon_\pm = \epsilon_0\left(1 + \chi_\pm^{NL}\right) \qquad n_\pm = \sqrt{\epsilon_\pm} \quad n_0 = \sqrt{\epsilon_0}. \tag{A.12}$$

Then, we obtain

$$n_\pm^2 = n_0^2\left(1 + \chi_\pm^{NL}\right) = n_0^2(1 + 6\chi_{1122}|E_\pm|^2 + 6(\chi_{1122} + \chi_{1221})|E_\mp|^2) \tag{A.13}$$

with its approximation

$$n_\pm \simeq n_0(1 + 3\chi_{1122}|E_\pm|^2 + 3(\chi_{1122} + \chi_{1221})|E_\mp|^2). \tag{A.14}$$

One infers the difference in optical indices:

$$\Delta n = n_+ - n_- = 3\chi_{1221}n_0(|E_-|^2 - |E_+|^2). \tag{A.15}$$

The solution for the electric field is (with c the light velocity)

$$\mathbf{E}(\mathbf{r}) = E_+(\mathbf{r})\hat{\sigma}_+ + E_-(\mathbf{r})\hat{\sigma}_- = A_+ \exp(\mathrm{i}n_+\omega z/c)\hat{\sigma}_+ + A_- \exp(\mathrm{i}n_-\omega z/c)\hat{\sigma}_- \tag{A.16}$$

which we can rewrite as

$$\mathbf{E}(\mathbf{r}) = (A_+ \exp(\mathrm{i}\Delta n\omega z/2c)\hat{\sigma}_+ + A_- \exp(-\mathrm{i}\Delta n\omega z/2c)\hat{\sigma}_-) \exp(\mathrm{i}[n_- + 1/2\Delta n]\omega z/c) \tag{A.17}$$

as a function of

$$k_n = (n_- + 1/2\Delta n)\frac{\omega}{c} \qquad \text{and} \qquad \theta_{NL} = \frac{\Delta n\omega z}{2c}. \tag{A.18}$$

We finally get

$$\mathbf{E}(\mathbf{r}) = (A_+ \exp(\mathrm{i}\theta_{NL})\hat{\sigma}_+ + A_- \exp(-\mathrm{i}\theta_{NL})\hat{\sigma}_-) \exp(\mathrm{i}k_n z). \tag{A.19}$$

Now, coming back to the realm of mechanics, we can define the analogue of the angular velocity of precession $\Lambda_p \simeq (R^2 - S^2)\omega_0$ of the conical pendulum in optics:

$$\Lambda_{NL} = \frac{\mathrm{d}\theta_{NL}}{\mathrm{d}z} = \frac{\Delta n\omega}{2c} = \frac{3}{2}\frac{\chi_{1221}n_0}{c}(|E_-|^2 - |E_+|^2)\omega, \tag{A.20}$$

and we recover the facts that the precession is proportional to both the area of the elliptic path described by the polarization vectors and the natural frequency of the phenomenon. In addition, either the circular (one of the two components $\hat{\sigma}_\pm$ vanishes) or linear polarization ($|E_-| = |E_+|$) avoids any precession as for the pendulum.

Appendix B

We follow the derivation of the Faraday effect given by Jonsson [10]. First, the plane wave is infinite which leads to $\mathbf{E}_\omega = \mathbf{E}_\omega(z)$.

One can show that the equation of propagation for light in an isotropic medium submitted to a static magnetic field ($\mathbf{B}_0 = B_0\mathbf{e}_z$) along the direction z of the wavevector is

$$\frac{\partial^2 \mathbf{E}_\omega}{\partial z^2} + \left(\frac{n\omega}{c}\right)^2 \mathbf{E}_\omega + \left(\frac{\omega}{c}\right)^2 \chi_{123}\mathbf{E}_\omega \times \mathbf{B}_0 = \mathbf{0} \tag{B.1}$$

where the last term is similar to the Coriolis-like term for the pendulum.

As for the nonlinear case described in the previous appendix, one decomposes the electric field in right and left polarizations:

$$\mathbf{E}_\omega = (E_+(z)\hat{\sigma}_+ + E_-(z)\hat{\sigma}_-) \exp(\mathrm{i}n\omega z/c). \tag{B.2}$$

One introduces this ansatz into the equation of propagation which gives

$$\frac{\partial E_\pm}{\partial z} = \pm\mathrm{i}\frac{\omega\gamma}{2nc}E_\pm \tag{B.3}$$

with $\gamma = \chi_{123}B_0$ and where we used the slowly varying envelope approximation (similar to $\ddot{A} \ll \dot{A}$ for the pendulum):

$$\left|\frac{\partial^2 \mathbf{E}_\pm}{\partial z^2}\right| \ll \left|\frac{2n\omega}{c}\frac{\partial \mathbf{E}_\pm}{\partial z}\right|. \tag{B.4}$$

We end up with

$$\mathbf{E}_\omega = E_+(0) \exp\left[\mathrm{i}\frac{\omega z}{c}\left(n + \frac{\gamma}{2n}\right)\right]\hat{\sigma}_+ + E_-(0) \exp\left[\mathrm{i}\frac{\omega z}{c}\left(n - \frac{\gamma}{2n}\right)\right]\hat{\sigma}_-. \tag{B.5}$$

We introduce the right and left index: $n_+ = n + \frac{\gamma}{2n}$ and $n_- = n - \frac{\gamma}{2n}$. Their difference becomes

$$\Delta n = n_+ - n_- = \frac{\gamma}{n} = \frac{\chi_{123}}{n} B_0. \tag{B.6}$$

One infers the angle of rotation

$$\Theta_B = \frac{\omega z}{c} \Delta n \tag{B.7}$$

and the associated spatial precession which is proportional to the magnetic field as expected with Faraday's law:

$$\Lambda_B = \frac{\mathrm{d}\Theta_B}{\mathrm{d}z} = \frac{\chi_{123}}{nc} \omega B_0. \tag{B.8}$$

Appendix C

The Lagrangian for a conical pendulum is

$$L^* = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{h}^2) - mgh. \tag{C.1}$$

We introduce dimensionless variables:

$$X = \frac{x}{l}, \qquad Y = \frac{y}{l}, \qquad H = \frac{h}{l}, \qquad t^* = t\sqrt{\frac{g}{l}}. \tag{C.2}$$

The spherical constraint leads to

$$H = 1 - \sqrt{1 - (X^2 + Y^2)} \simeq \frac{X^2 + Y^2}{2} + \frac{(X^2 + Y^2)^2}{8} \tag{C.3}$$

with $X \ll 1$ and $Y \ll 1$. Now, one can use the dimensionless complex position and Lagrangian,

$$Z = X + \mathrm{i}Y, \qquad L = \frac{L^*}{mgl} \tag{C.4}$$

with the Euler–Lagrange equation for the independent variables Z and $\overline{Z}$:

$$\frac{\partial L}{\partial \overline{Z}} = \frac{\partial}{\partial t^*} \frac{\partial L}{\partial \dot{\overline{Z}}} \tag{C.5}$$

with

$$\dot{H}^2 \simeq \tfrac{1}{4}(Z\dot{\overline{Z}} + \dot{Z}\overline{Z})^2, \tag{C.6}$$

where we kept the fourth-order term in the development of the height since it is necessary to balance the fourth-order term in the development of the vertical velocity. The other terms are of second order.

By recalling that $|Z|^2 = Z\overline{Z}$, we find

$$\frac{\partial L}{\partial \overline{Z}} = \frac{1}{4}\dot{Z}(\dot{Z}\overline{Z} + Z\dot{\overline{Z}}) - \frac{1}{2}Z - \frac{1}{4}\overline{Z}Z^2 \tag{C.7}$$

and

$$\frac{\partial L}{\partial \dot{\overline{Z}}} = \frac{1}{2}\dot{Z} + \frac{1}{4}Z(\dot{Z}\overline{Z} + Z\dot{\overline{Z}}) \tag{C.8}$$

then

$$\frac{\partial}{\partial t^*} \frac{\partial L}{\partial \dot{\overline{Z}}} = \frac{1}{2}\ddot{Z} + \frac{1}{4}(\dot{Z}^2\overline{Z} + \ddot{Z}|Z|^2 + Z\dot{Z}\dot{\overline{Z}} + 2|\dot{Z}|^2 Z + Z^2\ddot{\overline{Z}}). \tag{C.9}$$

Hence, we get the equation for Z:

$$\frac{1}{2}\left(1+\frac{|Z|^2}{2}\right)\ddot{Z}+\frac{1}{4}Z^2\ddot{\overline{Z}}+\frac{1}{2}Z+\frac{1}{4}|Z|^2Z+\frac{1}{2}|\dot{Z}|^2Z=0. \tag{C.10}$$

We can rewrite the last equation with its complex conjugate in terms of a system with the help of matrices:

$$M\begin{pmatrix}\ddot{Z}\\ \ddot{\overline{Z}}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}Z-\frac{1}{4}|Z|^2Z-\frac{1}{2}|\dot{Z}|^2Z\\ -\frac{1}{2}\overline{Z}-\frac{1}{4}|Z|^2\overline{Z}-\frac{1}{2}|\dot{Z}|^2\overline{Z}\end{pmatrix} \tag{C.11}$$

where

$$M=\begin{pmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{pmatrix}+\begin{pmatrix}\frac{1}{4}|Z|^2&\frac{1}{4}Z^2\\ \frac{1}{4}\overline{Z}^2&\frac{1}{4}|Z|^2\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{pmatrix}+O(Z^2). \tag{C.12}$$

Now, we use the following equations in order to invert the M matrix:

$$\begin{aligned}&M=\mu I_d+\nu N \qquad &&M^{-1}=\frac{1}{\mu}I_d+\nu P \qquad &&MM^{-1}=I_d\\ &P=-\frac{1}{\mu^2}N &&M^{-1}=\frac{1}{\mu}I_d-\frac{\nu}{\mu^2}N\end{aligned} \tag{C.13}$$

which gives

$$M^{-1}=\begin{pmatrix}2&0\\0&2\end{pmatrix}-\begin{pmatrix}|Z|^2&Z^2\\ \overline{Z}^2&|Z|^2\end{pmatrix}. \tag{C.14}$$

Finally, we obtain the equation of motion for the conical pendulum as a complex representation:

$$\ddot{Z}+Z-\tfrac{1}{2}|Z|^2Z+|\dot{Z}|^2Z=0 \tag{C.15}$$

where we have kept only the nonlinear terms up to third order in Z coming from the product,

$$M^{-1}\begin{pmatrix}-\frac{1}{2}Z\\ -\frac{1}{2}\overline{Z}\end{pmatrix}. \tag{C.16}$$

Appendix D

Rewriting the Lagrangian in spherical coordinates with dimensionless variables (time is measured in units of ω_0 and length in units of l) gives

$$L^*=\tfrac{1}{2}\dot{\theta}^2+\tfrac{1}{2}\sin^2\theta\dot{\phi}^2+\cos\theta \tag{D.1}$$

from which one easily extracts the equation of motion for θ:

$$\ddot{\theta}-\dot{\phi}^2\sin\theta\cos\theta+\sin\theta=0. \tag{D.2}$$

If $\ddot{\theta}=0$, the motion is spherical and one gets

$$\Omega(\theta)=\dot{\phi}=\frac{1}{\sqrt{\cos\theta}}. \tag{D.3}$$

With $\sin\theta=\sqrt{x^2+y^2}/l$, $\theta\ll 1$ and $X+\mathrm{i}Y=A\exp(\mathrm{i}t)$, one finds $|A|^2=X^2+Y^2\simeq\theta^2$.

The angular velocity of rotation can be approximated as

$$\Omega(\theta)\simeq\frac{1}{1-\theta^2/4}\simeq 1+\frac{\theta^2}{4} \tag{D.4}$$

which gives the part of the amplitude equation depending on A:

$$\Omega(|A|) \simeq 1 + \frac{|A|^2}{4}. \tag{D.5}$$

One can expect that the full amplitude equation with the B dependence should be written according to

$$\dot{A} = \mathrm{i}A\left(\tfrac{1}{4}|A|^2 + p|B|^q\right). \tag{D.6}$$

Now, with $A = B$, one must recover the equation for the nonlinear 'linear' (i.e. not conical) pendulum:

$$\dot{A} = -\frac{\mathrm{i}}{4}|A|^2 A. \tag{D.7}$$

One ends up with $p = -1/2$ $q = 2$ and the amplitude equation for the conical pendulum,

$$\dot{A} = \mathrm{i}A\left(\tfrac{1}{4}|A|^2 - \tfrac{1}{2}|B|^2\right). \tag{D.8}$$

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