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# Negative energy waves in a shear flow with a linear profile

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ABSTRACT

We present a derivation of the time averaged potential and kinetic energies for small-amplitude surface waves on a shear flow with constant vorticity. The effect of surface tension is also taken into consideration. It is demonstrated that the virial theorem of the energy equipartition between the potential and kinetic components is not valid in general for waves on a shear flow. We also show that waves with a negative energy may exist in a shear flow, and we find the condition for existence of such waves.


## 1. Introduction

The wave energy concept is of primary significance in the investigation of various problems of wave generation, propagation and absorption in hydrodynamic shear flows. Considering linear problems, we usually deal, however, with the 'quasi-energy' of monochromatic waves [1,2]. The physical meaning of the quasi-energy and its relation to the total energy of a hydrodynamic flow, including wave-induced flows, have been considered also in [3,4]. The important point following from the quasi-energy concept is that the wave energy may change its sign and become negative in the presence of a shear flow. When the negative energy waves (NEWs) exist, the total energy of a medium with waves becomes less than the energy of the medium without waves. So, no energy is needed to excite NEWs; instead, some energy can be extracted from the system by the excitation of such waves. The more energy is extracted, the more intensely NEWs are excited in the system.

The concept of waves possessing opposite sign energies in shear flows enables researchers to interpret various types of instabilities in hydrodynamics [3,4]. In particular, the well-known Kelvin–Helmholtz instability can be interpreted in terms of energy exchange between two modes — one of positive energy and another of negative energy [5,4]. It is clear that the NEWs may exist only in non-equilibrium media, in particular, in shear flows with a large stock of kinetic energy. In addition to hydrodynamic flows, there are many other examples of such media supporting NEWs; for example, beams of charged particles in a plasma, two-level atomic systems, etc.

The concept of negative energy waves was proposed by L. Chua as applied to waves in electron beams back in 1951 (see, e.g. [6]) and has since been widely used in hydrodynamics, plasma physics and electronics; numerous examples can be found in [7–10,5,11,4]. Nevertheless, this concept still causes certain difficulties to some researchers, therefore it seems reasonable to consider one more example of practical interest which demonstrates the possibility of existence of NEWs and represents a certain academic interest. We consider below surface gravity-capillary waves on a shear flow with the velocity profile linearly depending on the depth (the Couette-type shear flow with a constant vorticity).

There is another interesting aspect of the problem of wave propagation in shear flows. It is common knowledge that average potential and kinetic energies are equal in both surface and internal waves of small amplitude in a motionless fluid [12] by virtue of the virial theorem [13] for arbitrary mechanical systems. An average value of the Lagrangian function [14] that coincides with the difference between the kinetic and potential energies is equal to zero in this case. However, average kinetic and potential energies do not necessarily coincide in non-equilibrium media such as, for example, stratified fluids with shear flows. Actually, the kinetic energy density in a shear flow depends on the strength of the background

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flow (this will be specified below), whereas the potential energy is independent of a shear flow and determined solely by the displacement of the free surface (and isopycnic surfaces in a stratified fluid). In this case, the Lagrangian function is no longer determined by the difference between the kinetic and potential energies although the average value of the Lagrangian is still equal to zero. The latter may occur because the average Lagrangian of linear perturbations is proportional to the function $D(\omega; k)$, where $D(\omega; k)$ is proportional to the dispersion relation for small-amplitude perturbations [14,5]. Such systems are well known in mechanics and are referred to as non-natural systems [15].

In Ref. [16] was investigated the relation between the kinetic and potential energies for internal waves of infinitesimal amplitude at the interface between two infinitely thick layers of fluids with different densities moving with constant velocity with respect to each other (see also Sect. 1.9 in the book [4]). The problem was analysed both with and without surface tension effect between the layers. The explicit expressions for kinetic and potential energies have been derived and it has been shown that under a certain condition the wave energy may become negative.

The wave energy for surface gravity-capillary waves on a uniform flow has been also calculated by Dysthe [17] for a fluid of a finite depth. The analysis of results obtained can reveal the existence of NEWs in such system, but this issue has not been considered in Dysthe's Lecture Notes.

In the recent publication by Ellingsen and Brevik [18] the wave energy of surface gravity waves on a shear flow with linear velocity profile has been calculated for a fluid of a finite depth. But the authors obtained an incorrect result (the source of the error will be elucidated below) and did not discuss the possibility of existence of NEWs.

Below we calculate the total energy of small-amplitude surface gravity-capillary waves on a shear flow with a linear velocity profile in a fluid of a finite depth. Then we consider the relationship between the kinetic and potential energies for such waves and show that they are not equal in general. With the surface tension effect taken into account we derive the condition when NEWs appear in the system. This problem represents not only an academic interest, but may be important for practical applications. The results obtained can be further developed in application to the study of wave blocking phenomena when a shear flow gradually varies in the horizontal direction.

## 2. Problem statement and the dispersion relation

Let us consider one-dimensional wave propagation on a surface of moving water of a finite depth $h$. We assume that the velocity profile of the water flow $U(z)$ is a linear function of the depth $U(z) = U_0 + \alpha z$. Here $U_0$ is the water speed at the free surface, and $\alpha$ characterises the vorticity of the background flow. When the fluid velocity vanishes at the bottom, then $\alpha = U_0/h$, but in general $\alpha$ may be an independent parameter; in particular, putting $\alpha = 0$ we obtain the uniform current without vorticity. The shear flow vorticity can be controlled in the finite depth fluid with the help of a movable bottom, for example, by using a rubber conveyor. It is assumed that the axis $z$ is directed upward with a zero at the unperturbed water surface. For certainty we suppose that $U_0 > 0$, i.e. the background flow at the water surface is co-directed with the axis $x$; in other words, the velocity vector is $\mathbf{U}(z) = U_0(z)\hat{\mathbf{i}}$. The sketch of the flow considered here is shown in Fig. 1.

The dispersion relation for surface waves in water with a current linearly varying with depth has been derived both for pure gravitational waves [19–22] and for capillary–gravity waves [23,24,18]. We will re-derive it below in the form convenient for our analysis. The main aim of this paper is to derive in the linear approximation the wave energy of surface gravity-capillary waves

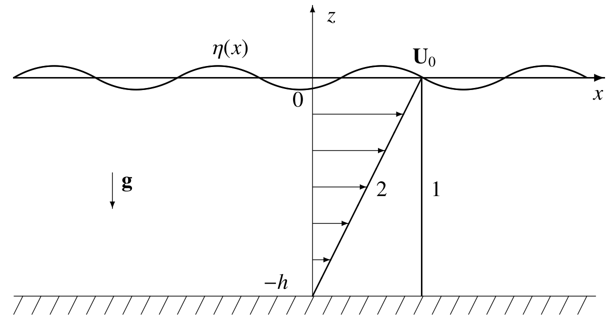


**Fig. 1.** Sketch of the fluid flow in the reference coordinate frame associated with the immovable bottom. Line 1 depicts the velocity with the uniform profile and line 2 the velocity with the linear profile.

propagating on a shear flow and demonstrate some interesting nontrivial features related to it. In particular, we show that under certain conditions the wave energy may become negative which indicates that the flow may become unstable. The existence of negative energy waves in uniformly moving media is well-known (see the references listed above), but to the best of our knowledge, the influence of the basic flow vorticity on the wave energy has not been studied thus far. In our study we obtain the general results which account for the arbitrary flow vorticity and naturally reduce to the case of a uniformly moving fluid. We calculate explicitly the ratio of kinetic to potential energies for a linear wave and analyse in detail its dependence on the flow intensity, vorticity and surface tension. We also show that the virial theorem of the energy equipartition between these two energy components is not valid for waves on a linear shear flow.

### 2.1. Basic equations and derivation of the dispersion relation

For our purposes it is convenient to present the Euler equation in the Helmholtz form [25]:

$$\frac{\partial(\mathrm{curl}\,\mathbf{v})}{\partial t} = (\mathrm{curl}\,\mathbf{v}\cdot\nabla)\,\mathbf{v} - (\mathbf{v}\cdot\nabla)\,\mathrm{curl}\,\mathbf{v}. \tag{1}$$

In a two-dimensional flow when all variables depend on two spatial coordinates $x$ and $z$, the curl of any vector is perpendicular to the $x, z$-plane, whereas operator nabla has components in that plane. Therefore the first term on the right-hand side of Eq. (1) is zero, and the equation reads

$$\frac{\partial(\mathrm{curl}\,\tilde{\mathbf{v}})}{\partial t} = -(\mathbf{v}\cdot\nabla)\,(\mathrm{curl}\,\mathbf{U} + \mathrm{curl}\,\tilde{\mathbf{v}}), \tag{2}$$

where $\tilde{\mathbf{v}}$ is the perturbation of the velocity field, and $\mathbf{U}(z)$ is the background flow with the linear profile shown in Fig. 1; its curl is constant and directed perpendicular to the $x, z$-plane, $\mathrm{curl}\,\mathbf{U} = \alpha\,\hat{\mathbf{j}}$ (the flow with constant vorticity). In this case the time derivative of the basic vorticity vanishes, and the velocity perturbation remains potential if it was potential initially. This allows us to consider the potential velocity perturbation $\tilde{\mathbf{v}} \equiv (u, v) = \nabla\varphi$ superimposed on the background flow of constant vorticity. Note that this is the only case of vortical flow which is compatible with potential perturbations [26]. Then we have the following equation for the velocity potential in the entire fluid domain:

$$\Delta\varphi = 0. \tag{3}$$

The boundary conditions are conventional: there is no water flow through the rigid bottom, therefore

$$\frac{\partial\varphi}{\partial z} = 0 \quad \text{at } z = -h. \tag{4}$$

Then there is a kinematic boundary condition at the free surface $\eta(x,t)$ which reads in the linear approximation as:

$$\frac{\partial \eta}{\partial t} + U_0 \frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial z} \quad \text{at } z = 0. \tag{5}$$

The dynamic boundary condition at the free surface can be readily derived in the linear approximation. To this end let us present the Euler equation in the Lamb form [25]:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2} + \frac{P}{\rho} + gz\right) = \mathbf{v} \times \operatorname{curl} \mathbf{v}, \tag{6}$$

where $P(z)$ is the pressure, $\rho$ is water density, and $g$ is the acceleration due to gravity.

Consider now the $x$ component of this equation at the surface $z = 0$ and substitute into it $\mathbf{v} = U(z)\hat{\mathbf{i}} + \nabla\varphi$, where $U(z) = U_0 + \alpha z$, and the pressure $P(0) = P_a + \rho g \eta - \sigma\left(\partial^2\eta/\partial x^2\right)$ at $z = 0$, where $P_a$ is the atmospheric pressure which is assumed to be a constant, and $\sigma$ is the surface tension coefficient. Then we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t} + U_0 \frac{\partial \varphi}{\partial x} + g\eta - \gamma \frac{\partial^2 \eta}{\partial x^2}\right) = -\alpha \frac{\partial \varphi}{\partial z} \quad \text{at } z = 0, \tag{7}$$

where $\gamma = \sigma/\rho$ is the normalised surface tension. This is nothing but the Bernoulli equation for a potential motion in the linear shear flow (cf. [18]).

Looking for an elementary solution to the boundary value problem in the form

$$\eta = A \exp[\mathrm{i}(\omega t - \hat{\mathbf{i}}{\cdot}\mathbf{k}\, x)],$$

$$\varphi = B \cosh|k|(z + h) \exp[\mathrm{i}(\omega t - \hat{\mathbf{i}}{\cdot}\mathbf{k}\, x)],$$

we obtain from Eq. (5):

$$B = \mathrm{i}A \frac{\omega - \mathbf{U}{\cdot}\mathbf{k}}{|k| \sinh|k|h}, \tag{8}$$

where $\omega$ is the frequency of infinitesimal amplitude surface wave, $\mathbf{k} = k\hat{\mathbf{i}}$ is the wavenumber which can be either parallel or anti-parallel to the vector $\mathbf{U}$ depending on the sign of the parameter $k$.

Then, from Eq. (7) we derive the dispersion relation [23,24,18]:

$$(\omega - \mathbf{U}{\cdot}\mathbf{k})^2 + \alpha \frac{\mathbf{U}{\cdot}\mathbf{k}}{U_0|k|} \tanh(|k|h)(\omega - \mathbf{U}{\cdot}\mathbf{k}) - \left(g + \gamma k^2\right)|k| \tanh(|k|h) = 0. \tag{9}$$

Solution to this quadratic equation can be presented in the form:

$$\omega^{\pm} = \mathbf{U}{\cdot}\mathbf{k}\left(1 - \frac{\alpha}{2}\frac{\tanh|k|h}{U_0|k|}\right) + \sqrt{\left(\frac{\alpha}{2}\tanh|k|h\right)^2 + \left(g + \gamma k^2\right)|k| \tanh|k|h}. \tag{10}$$

For the better understanding of what this equation describes, we present below a graphical analysis of its dimensionless form:

$$\tilde{\omega}^{\pm} = \left(\hat{\mathbf{U}}_0{\cdot}\hat{\mathbf{k}}_0\right)(\mathrm{Fr}\,|\kappa| - \Omega \tanh|\kappa|) + \sqrt{(\Omega \tanh|\kappa|)^2 + \left(1 + \mathrm{S}\kappa^2\right)|\kappa| \tanh|\kappa|} \tag{11}$$

where $\tilde{\omega} = \omega\sqrt{h/g}$, $\kappa = kh$, $\mathrm{Fr} = U_0/\sqrt{gh}$ is the Froude number, $\Omega = \alpha h/(2\sqrt{gh})$ is the normalised vorticity of the basic flow, and $\mathrm{S} = \gamma/(gh^2) = \sigma/(\rho g h^2)$ is the normalised surface tension, and $\hat{\mathbf{U}}_0$ and $\hat{\mathbf{k}}_0$ are the unit vectors of the flow velocity and wavenumber, correspondingly. (Notice from the first term in Eq. (10), in the shallow-water limit, $\tanh|k|h \to |k|h$, that it may be reasonable to define the effective Froude number for shear flows as the ratio of the mean speed (rather than maximal speed $U_0$) over the speed of long waves. In such a case the effective Froude number for a shear flow with constant vorticity shown in Fig. 1 (see line 2) would be $\mathrm{Fr}_{eff} = U_0/2\sqrt{gh} = \mathrm{Fr}/2$. Here we use, however, a more traditional definition of Froude number.)

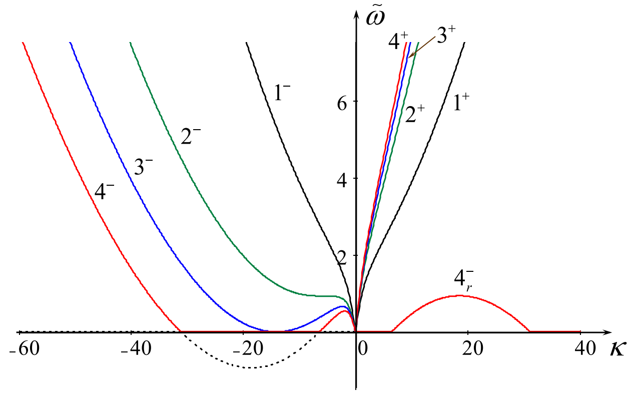


**Fig. 2.** (Colour online) Dependence of dimensionless wave frequency on the dimensionless wavenumber at different Froude numbers as per Eq. (11) with $\Omega = 0$ and $\mathrm{S} = 0.005$.

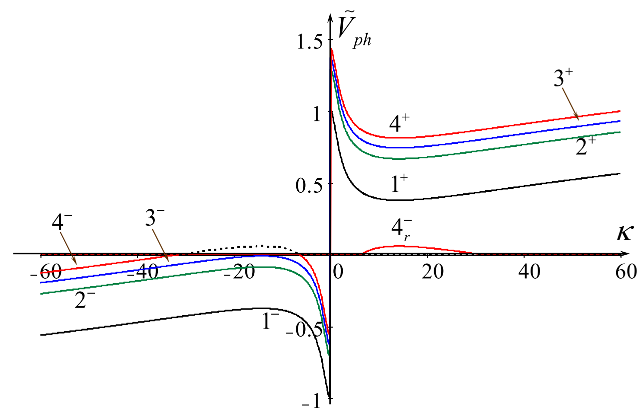


**Fig. 3.** (Colour online) Dependence of dimensionless phase speed on the dimensionless wavenumber at different Froude numbers as per Eq. (11) with $\Omega = 0$ and $\mathrm{S} = 0.005$.

For simplicity we first neglect the flow vorticity by setting $\Omega = 0$, and we put for definiteness $\mathrm{S} = 0.005$. Then in Fig. 2 we show the evolution of dispersion curves described by Eq. (11) when the Froude number increases.

Lines $1^+$–$4^+$ represent dispersion relations $\tilde{\omega}^+(\kappa)$ for waves co-propagating with the current so that $\hat{\mathbf{U}}_0{\cdot}\hat{\mathbf{k}}_0 = 1$; their phase speeds ($\tilde{V}_{ph} \equiv \tilde{\omega}/\kappa$) are positive for any Froude number (see Fig. 3).

For the counter-current propagating waves $\hat{\mathbf{U}}_0{\cdot}\hat{\mathbf{k}}_0 = -1$ the situation is a bit more interesting. When $\mathrm{Fr} = 0$, line $1^-$ is just a mirror reflection of line $1^+$ about the vertical axis in Fig. 2; it describes surface-gravity waves propagating to the left with negative phase speed (cf. lines $1^+$ and $1^-$ in Fig. 3). When the Froude number increases, line $1^-$ transforms into line $2^-$, and then, when the Froude number attains the critical value $\mathrm{Fr} = \mathrm{Fr}_c$, it transforms into line $3^-$ as shown in Fig. 2. At this critical value the dispersion line touches the $\kappa$-axis, and there is a finite $\kappa$ for which $\tilde{\omega}^-$ becomes equal to zero together with the phase speed. Notice that at $\mathrm{Fr} < \mathrm{Fr}_c$, the phase speeds are negative for $\kappa < 0$. This means that the corresponding waves propagate against the current (see lines $2^-$ and $3^-$ in Fig. 3).

When Fr becomes greater than $\mathrm{Fr}_c$, the wave frequency formally becomes negative for a certain interval of wavenumbers (see lines $4^-$ and its dashed portion in Fig. 2); the corresponding phase speed on that interval becomes positive (because $\tilde{\omega} < 0$ and $\kappa < 0$

simultaneously). This means that the current becomes so strong that it pulls these waves in the direction of the flow. As a result the waves move downstream relative to the immovable observer. In the immovable coordinate frame such co-current propagating waves have positive frequencies and wavenumbers; their speeds are shown by line $4_r^-$ in Fig. 2. Formally the transformation from the dashed portion of line 4 to the corresponding solid line $4_r^-$ follows from the invariance of the dispersion relation (10) with respect to the transformation $\omega^{\pm} \to -\omega^{\pm}$ and $\mathbf{k} \to -\mathbf{k}$ with simultaneous change of sign in front of the square root.

Usually the appearance of negative frequencies in a wave system signals that potentially unstable NEWs exist in such a system [7–10,5,11,3,4]. This is also the case for surface waves on a uniform current ($\Omega = 0$) considered here. However, in the presence of vorticity (when $\Omega \neq 0$) the situation becomes more complicated, and the condition for existence of NEWs depends on the vorticity. Below we consider this issue in detail.

## 3. Wave energy

Following [16,4,17], let us calculate the wave energy of small perturbations averaged over the wave period on the basis of the solution derived in the previous section. The total wave energy can be presented as the sum of the potential energy $P$ and kinetic energy $K$: $E = P + K$. In the linear approximation the wave energy is proportional to the squared wave amplitude, therefore in the following calculations we will retain only those terms which are quadratic in the wave amplitude. The potential energy does not depend on a flow; it is determined entirely by the deflection of the free surface from the equilibrium position $z = 0$:

$$P = \left\langle \frac{\rho g}{2}\eta^2 + \frac{\sigma}{2}\left(\frac{\partial \eta}{\partial x}\right)^2 \right\rangle = \frac{\rho A^2}{4}\left(g + \gamma k^2\right), \tag{12}$$

where angular brackets stand for averaging over the wave period $T$:

$$\langle f(t) \rangle \equiv \frac{1}{T}\int_0^T f(t)\, dt. \tag{13}$$

The kinetic energy of wave motion can be determined as the kinetic energy of a moving fluid with a wave perturbation minus the kinetic energy of the same moving fluid without the perturbation:

$$\begin{aligned} K &= \frac{\rho}{2}\left\langle \int_{-h}^{\eta} \left\{[u + U(z)]^2 + v^2 - U^2(z)\right\} dz \right\rangle \\ &\approx \frac{\rho}{2}\left\langle \int_{-h}^{0} \left(u^2 + v^2\right) dz \right\rangle + \rho \left\langle \int_{-h}^{\eta} uU(z)\, dz \right\rangle. \end{aligned} \tag{14}$$

The first integral in the last expression represents an essentially positive quantity, whereas the second integral can be of either sign. So, we can replace the upper limit of integration in the first integral by zero, because the integrand is the quantity which is already quadratic in the wave amplitude, since $u \sim v \sim A$ (see Eqs. (16) and (17)). However, this is not the case for the second integral as it contains a product $uU(z)$ which is of first order in the wave amplitude. This integral can be presented as the sum

$$\begin{aligned} &\rho \left\langle \int_{-h}^{0} uU(z)\, dz \right\rangle + \rho \left\langle \int_{0}^{\eta} uU(z)\, dz \right\rangle \\ &\quad \approx \rho U(0) \left\langle \eta(x,t) u(x,0,t) \right\rangle. \end{aligned} \tag{15}$$

The first term on the left hand side gives zero after averaging on time as it is a sinusoidal function of time (it is linear in the perturbation). The second integral can be approximately calculated with the help of the Lagrange mean value formula. As we consider linear perturbations of infinitesimally small amplitude, the variation with $z$ of $U_0$ and $\partial\varphi/\partial x$ on the interval $[0, \eta]$ is insignificant. Using the solution derived in Section 2 we can readily calculate all integrals. We present first the perturbed velocity field in the real form:

$$\begin{aligned} u(x,z,t) &= \mathrm{Re}\left(\frac{\partial \varphi}{\partial x}\right) \\ &= A\,\hat{\mathbf{i}}\cdot\mathbf{k}\frac{\omega - \mathbf{U}\cdot\mathbf{k}}{|k|\sinh|k|h}\cosh|k|(z+h)\cos(\omega t - \hat{\mathbf{i}}\cdot\mathbf{k}\,x), \end{aligned} \tag{16}$$

$$\begin{aligned} v(x,z,t) &= \mathrm{Re}\left(\frac{\partial \varphi}{\partial z}\right) \\ &= -A\frac{\omega - \mathbf{U}\cdot\mathbf{k}}{\sinh|k|h}\sinh|k|(z+h)\sin(\omega t - \hat{\mathbf{i}}\cdot\mathbf{k}\,x). \end{aligned} \tag{17}$$

Substituting this into Eqs. (14) and (15) we obtain for the kinetic energy averaged over the wave period:

$$K = \frac{\rho A^2}{4}\frac{\omega^2 - U_0^2 k^2}{|k|\tanh|k|h}. \tag{18}$$

Note that in [18] the authors have calculated the kinetic energy of pure gravity surface waves on a linear shear flow. However, they obtained an incorrect result because they mistakenly assumed that the last integral in Eq. (14) depends linearly in the wave perturbation and therefore vanishes after averaging over time. As has been shown above, that integral consists of two parts (see Eq. (15)), one of which indeed depends linearly on the wave perturbation, while the other depends quadratically on the perturbation (via the upper limit!) and contributes to the kinetic energy. This agrees with the earlier findings for kinetic energies of surface and internal waves in uniform flows [16,4,17].

Using the dispersion relation (10) we can present the kinetic energy for co-propagating waves $K^+$ and counter-propagating waves $K^-$ in the following form

$$\begin{aligned} K^{\pm} = \frac{\rho A^2}{4}\Bigg\{ & g + \gamma k^2 + \alpha\left(U_0 - \frac{\alpha}{2}\frac{\tanh|k|h}{|k|}\right) \\ & \times \left[\left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)\sqrt{1 + \frac{4|k|\left(g + \gamma k^2\right)}{\alpha^2 \tanh|k|h}} - 1\right]\Bigg\}. \end{aligned} \tag{19}$$

As one can see from the expressions for the potential and kinetic energies, Eqs. (12) and (19), they are not equal in general. They become equal in the absence of the background flow in accordance with the virial theorem [13] and at some particular values of the wavenumber when $U_0 \neq 0$.

### 3.1. The relationship between the kinetic and potential energies

Let us consider the ratio of the kinetic and potential energies and analyse it as a function of wavenumber and other parameters in the dimensionless variables defined after Eq. (11):

$$\begin{aligned} R^{\pm} \equiv \frac{K^{\pm}}{P} = 1 + 2\frac{\Omega}{1 + S\kappa^2}\left(\mathrm{Fr} - \Omega\frac{\tanh|\kappa|}{|\kappa|}\right) \\ \times \left[\left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)\sqrt{1 + \frac{|\kappa|\left(1 + S\kappa^2\right)}{\Omega^2\tanh|\kappa|}} - 1\right]. \end{aligned} \tag{20}$$

As one can see, the energy ratio depends on many parameters, $\kappa$, Fr, $\Omega$, and S. To simplify the situation let us neglect first the basic flow vorticity by putting $\Omega = 0$. Then we obtain:

$$R^{\pm} = 1 + 2\left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)\mathrm{Fr}\sqrt{\frac{|\kappa|}{\left(1 + S\kappa^2\right)\tanh|\kappa|}}. \tag{21}$$

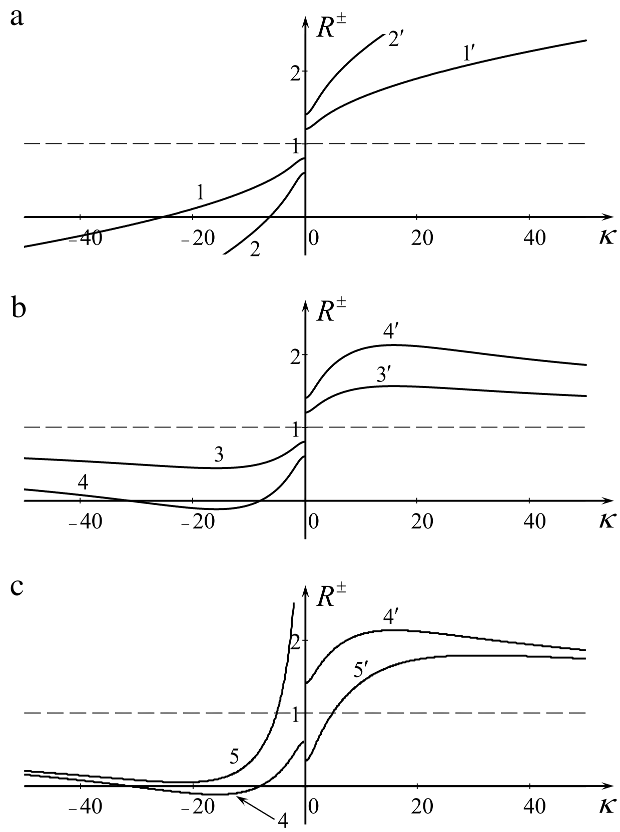


**Fig. 4.** Dependences of the energy ratio on the dimensionless wavenumber. Frame (a) — no vorticity ($\Omega = 0$) and no surface tension (S = 0); lines 1 and 1′ pertain to Fr = 0.1, lines 2 and 2′ — to Fr = 0.2. Frame (b) — no vorticity ($\Omega = 0$), but nonzero surface tension (S = 0.04); lines 3 and 3′ pertain to Fr = 0.1, lines 4 and 4′ — to Fr = 0.2. Frame (c) demonstrates the effect of vorticity at constant Froude number (Fr = 0.2) and surface tension (S = 0.04); lines 4 and 4′ are the same as in frame (b) when $\Omega = 0$; lines 5 and 5′ pertain to $\Omega = 1$.

Plots of $R^{\pm}(\kappa)$ for two values of Froude number are shown in Fig. 4 (see frames (a) and (b)) — cf. with the plots of $R^{\pm}$ for internal waves on the interface between two moving layers [16,4].

There are several specific features which are worthy of discussion. First of all, note that the branches 1′ and 1, as well as 2′ and 2 for positive and negative wavenumbers (i.e. for co- and counter-flow propagating waves) do not meet each other at $\kappa = 0$, because $\hat{\mathbf{k}}_0$ changes sign, being a unit vector. So that $R^+ - R^- = 4$Fr. Secondly, the kinetic energy of counter-current propagating waves is less than the potential energy; whereas it is greater than the potential energy for the co-current propagating waves. Thirdly, the energy ratio may become negative for counter-current propagating waves when the Froude number is sufficiently large. This can occur when the kinetic energy becomes negative (the potential energy is always positive as it follows from Eq. (12)); the condition for this is

$$\mathrm{Fr} > \frac{\mathrm{Fr}_{c1}}{2}, \quad \text{where } \mathrm{Fr}_{c1} = \sqrt{\left(1 + \mathrm{S}\kappa^2\right)\frac{\tanh|\kappa|}{|\kappa|}}. \tag{22}$$

When the surface tension is neglected, S = 0, then Eq. (22) has only one real root $\kappa(\mathrm{Fr})$ for any Fr $< 1/2$; the kinetic energy is negative for $\kappa < -\kappa_c$, where $\kappa_c$ can be determined from Eq. (22) for a fixed value of the Froude number. But if Fr $> 1/2$, then the kinetic energy is negative for all counter-current propagating waves, including infinitely long waves.

With the surface tension taken into consideration, the kinetic energy may become negative only when the Froude number is sufficiently large Fr $> \mathrm{Fr}_{min}$ (see line 4 in Fig. 4(b)); the threshold value $\mathrm{Fr}_{min}$ depends on the surface tension parameter S. It can be determined from Eq. (22) and presented in the parametric form:

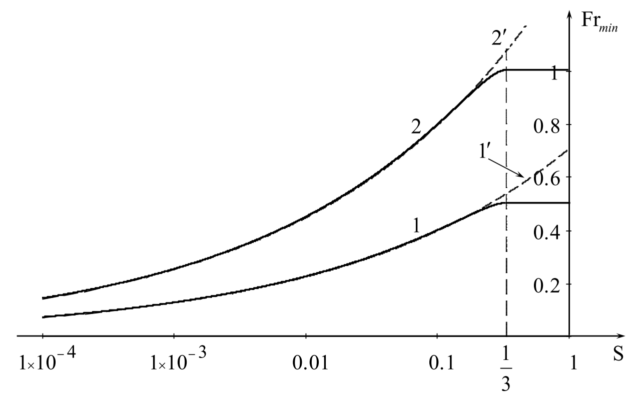


**Fig. 5.** Dependences of the minimum Froude numbers on the surface tension parameter when (i) the kinetic energy vanishes and then becomes negative (line 1) and (ii) when the total energy vanishes and then becomes negative (line 2). Dashed vertical line shows the maximum possible value of $\mathrm{S}_{max} = 1/3$.

$$\mathrm{S}(\kappa) = \frac{1}{\kappa^2} \cdot \frac{\tanh|\kappa| - |\kappa|(1 - \tanh^2|\kappa|)}{\tanh|\kappa| + |\kappa|(1 - \tanh^2|\kappa|)}, \tag{23}$$

$$\mathrm{Fr}_{min}(\kappa) = \frac{\tanh|\kappa|}{\sqrt{2|\kappa|\left[\tanh|\kappa| + |\kappa|(1 - \tanh^2|\kappa|)\right]}}. \tag{24}$$

The dependence $\mathrm{Fr}_{min}(\mathrm{S})$ is shown in Fig. 5. Asymptotically, when $\kappa \to -\infty$ ($\mathrm{S} \to 0$), it reduces to $\mathrm{Fr}_{min}(\mathrm{S}) = \sqrt[4]{\mathrm{S}/4}$ (see line 1′ in Fig. 5); in the dimensional variables this dependence reads $(U_0)_{min} = \sqrt[4]{\sigma g/4\rho}$. As will be shown below, the total energy becomes negative at $2\mathrm{Fr}_{min}(\mathrm{S}) = \sqrt[4]{4\mathrm{S}}$ or in the dimensional variables at $U_0 = \sqrt[4]{4\sigma g/\rho}$, which is nothing but the minimum phase speed of gravity-capillary waves in a deep water without a background flow.

In another limiting case when $|\kappa| \to 0$ ($\mathrm{S} \to 1/3$) the dependence $\mathrm{Fr}_{min}(\mathrm{S})$ asymptotically reduces to $\mathrm{Fr}_{min}(\mathrm{S}) = (67 + 195\mathrm{S} + 45\mathrm{S}^2 - 675\mathrm{S}^3)/224$. When S reaches 1/3, the kinetic energy vanishes at $\kappa = 0_-$ and Fr = 0.5 (the function $R^-(\kappa)$ has a minimum at this point). For all greater values of S the function $R^-(\kappa)$ has a minimum at $\kappa = 0_-$ and Fr > 0.5.

To illustrate the effect of vorticity on the energy ratio we present in Fig. 4(c) two lines – one for the case when there is no vorticity, $\Omega = 0$, and another one for the case when there is a vorticity, $\Omega = 1$. As one can see, the vorticity effect is similar, to a certain extent, to the increase of surface tension. It reduces the effect of current and prevents the appearance of negative values of kinetic energy. The maximal value of $R^-_{max} = 1 - 2(\mathrm{Fr} - \Omega)\left(\sqrt{\Omega^2 + 1} + \Omega\right)$ is attained at $\kappa = 0_-$. Meanwhile, the minimum value of $R^+_{min} = 1 + 2(\mathrm{Fr} - \Omega)\left(\sqrt{\Omega^2 + 1} - \Omega\right)$ is attained at $\kappa = 0_+$, but it always remains positive.

### 3.2. Negative energy waves

Let us now calculate the total period-averaged wave energy $E = P + K$:

$$E^{\pm} = \frac{\rho A^2}{2}\left\{g + \gamma k^2 + \frac{\alpha}{2}\left(U_0 - \frac{\alpha}{2}\frac{\tanh|k|h}{|k|}\right) \times \left[\left(\hat{\mathbf{U}}_0 \cdot \hat{\mathbf{k}}_0\right)\sqrt{1 + \frac{4|k|\left(g + \gamma k^2\right)}{\alpha^2 \tanh|k|h}} - 1\right]\right\}. \tag{25}$$

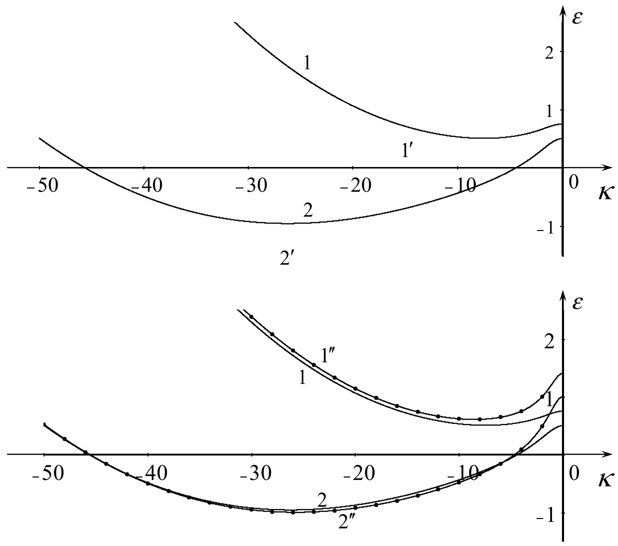


**Fig. 6.** Dependence of normalised wave energy on different dimensionless parameters. In frame (a) $\Omega = 0$; solid lines pertain to $S = 0.005$, whereas dashed lines pertain to $S = 0$. Lines 1 and 1′ are plotted for $Fr = 0.25$; lines 2 and 2′ — for $Fr = 0.5$. Frame (b): lines 1 and 2 are the same as in frame (a); dotted lines 1″ and 2″ are plotted for $\Omega = 0.5$, $S = 0.005$, but for different Froude numbers: line 1″ — for $Fr = 0.25$, and line 2″ — for $Fr = 0.5$.

We again present this quantity in dimensionless variables:

$$\mathcal{E}^{\pm} \equiv \frac{2E^{\pm}}{\rho g A^2} = 1 + S\kappa^2 + \Omega\left(Fr - \Omega\frac{\tanh|\kappa|}{|\kappa|}\right) \times \left[\left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)\sqrt{1+\frac{\kappa\left(1+S\kappa^2\right)}{\Omega^2\tanh\kappa}} - 1\right]. \quad (26)$$

Graphics of $\mathcal{E}^{-}(\kappa)$ for counter-flow propagating waves with different values of Fr, $\Omega$ and S are shown in Fig. 6.

Using the dispersion relation (10), the expression for the wave energy can also be presented in the form

$$E^{\pm} = \frac{\rho A^2}{2|k|\tanh|k|h}\left(\omega^{\pm} - \mathbf{U}\cdot\mathbf{k}\right)\left[\omega^{\pm} + \left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)\frac{\alpha}{2}\tanh|k|h\right], \quad (27)$$

or in the dimensionless form:

$$\mathcal{E}^{\pm} = \frac{\left[\tilde{\omega}^{\pm} - \left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)Fr\,|\kappa|\right]\left[\tilde{\omega}^{\pm} + \left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)\Omega\tanh|\kappa|\right]}{|\kappa|\tanh|\kappa|}. \quad (28)$$

Eq. (27) can be compared with what was derived by Dysthe [17] in the case of a uniform flow with $\alpha = 0$ ($\Omega = 0$).

As one can see from Eq. (26) and Fig. 6, not only the kinetic energy, but the total energy may also be negative. From the analysis of Eq. (28) it follows that the wave energy may be negative only for counter-current propagating waves when $\tilde{\omega}^{-} + Fr\,|\kappa| > 0$ and $\tilde{\omega}^{-} - \Omega\tanh|\kappa| < 0$, where $\kappa < 0$. The former condition is fulfilled for any negative $\kappa$, whereas the latter condition by means of the dispersion relation Eq. (11) for $\tilde{\omega}^{-}$ reduces to:

$$Fr > Fr_c \equiv \sqrt{\left(1 + S\kappa^2 + \Omega^2\frac{\tanh\kappa}{\kappa}\right)\frac{\tanh\kappa}{\kappa}}. \quad (29)$$

Thus, one can see that when waves propagate on a shear flow, the wave energy becomes negative not together with the frequency, but when $\tilde{\omega}^{-} < \Omega\tanh|\kappa|$. This circumstance was not previously known.

The concept of negative energy waves is very important in the general wave theory, because existence of such waves in a physical system signals a potential instability of the system. This issue has been discussed in many books and reviews, see, for instance, [7–9,5,10,11,3,4] and references therein. In particular, the well-known dissipative and radiative instabilities of waves in shear flows (see the literature cited above) are associated with the NEWs. When there is an appropriate energy sink in the system, NEWs exponentially grow with time in the linear approximation due to conversion of the kinetic energy of a shear flow into the wave energy of small perturbations. Note, however, that the existence of NEWs does not imply the presence of an instability, in general. They can grow only when there is an appropriate sink of energy. The wave energy depends on the choice of a coordinate frame, so that it may be negative in one coordinate frame, but positive in another. However, the dissipative function changes accordingly in the transition from one coordinate frame to another, so that the fact of stability or instability is invariant and does not depend on the choice of coordinate system. The details can be found in [5,10,11,3,4].

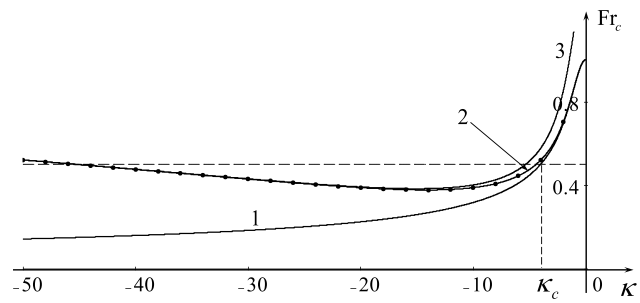


**Fig. 7.** Dependences of the critical Froude number as functions of $\kappa$ for several values of $\Omega$ and S for counter-flow propagating surface waves. Line 1 pertains to the case $\Omega = 0$, $S = 0$; line 2 with dots pertains to $\Omega = 0$, $S = 0.005$; line 3 pertains to $\Omega = 1$, $S = 0.005$.

In the particular case of a uniform flow ($\Omega = 0$) Eq. (29) reduces to $Fr_{c1}$ as defined in Eq. (22). The minimum value of $Fr_c$ in the deep-water limit when $\kappa \gg 1$ is $(Fr_c)_{min} = \sqrt[4]{4S}$. In the same limiting case, the minimum value of $Fr_c$ for the flow with a constant non-zero vorticity ($\Omega \neq 0$) is a more complicated function of S and $\Omega$. Replacing $\tanh\kappa$ by 1 in Eq. (29) for $Fr_c$ and solving for the stationary point for $\kappa$, we obtain

$$\kappa_{st}(S,\Omega) = \mathrm{Re}\left(\frac{3 + \left\{3\sqrt{S}\left[9\Omega^2 + \sqrt{3\left(27\Omega^4 - 1/S\right)}\right]\right\}^{2/3}}{\left\{(9S)^2\left[9\Omega^2 + \sqrt{3\left(27\Omega^4 - 1/S\right)}\right]\right\}^{1/3}}\right) \approx \frac{1 + \Omega^2\sqrt{S} - 3\Omega^4 S/2}{\sqrt{S}}. \quad (30)$$

The latter approximation presumes that the quantity $\Omega^2\sqrt{S}$ is small $\left(\Omega^2\sqrt{S} \ll 1\right)$.

Substituting $\kappa_{st}(S,\Omega)$ into the equation for $Fr_c$, we obtain approximately:

$$(Fr_c)_{min} \approx \sqrt[4]{4S} + \frac{\sqrt{2}}{4}\Omega^2 S^{3/4} - \frac{9\sqrt{2}}{32}\Omega^4 S^{5/4}. \quad (31)$$

Plots of the critical Froude number as functions of $\kappa$ are shown in Fig. 7 for several values of $\Omega$ and S.

As one can see from Eq. (29), if there is no surface tension, the dependence of $Fr_c$ on $\kappa$ is monotonic which means that for each given value of $Fr < \sqrt{1+\Omega^2}$ there is only one root $\kappa_c < 0$. For all $\kappa \in (-\infty, \kappa_c)$ there are NEWs in the system. This is illustrated by line 1 in Fig. 7. In particular, when $Fr > \sqrt{1+\Omega^2}$, then $\kappa_c = 0$, and all counter-current propagating waves become NEWs.

When the surface tension is taken into consideration, then NEWs may exist if $\mathrm{Fr} > (\mathrm{Fr}_c)_{min}$ which depends on S and $\Omega$. In this case, NEWs may exist only in the finite range of $\kappa$ between two intersections of line 2 with dashed horizontal line shown in Fig. 7. In particular, if $\mathrm{Fr} > \sqrt{1+\Omega^2}$, then NEWs range from a certain value $\kappa_m < \kappa_c$ ($|\kappa_m| \gg |\kappa_c|$) to zero. The influence of vorticity is again similar to the increase of surface tension (see line 3 in Fig. 7) — the range of NEWs becomes narrower and the threshold value of their appearance increases. However, unlike the surface tension effect, the vorticity cannot prevent the appearance of NEWs.

In general, the minimal Froude number leading to the appearance of NEWs can be found from Eq. (29) in the parametric form. It can be readily shown that NEWs appear when the kinetic energy is negative and two times greater in absolute value than the potential energy.

## 4. Discussion and conclusion

In this paper we have presented a detailed analysis of wave energy for infinitesimal amplitude surface waves on a shear flow with a linear profile. It has been shown that at a certain velocity of the background flow the wave energy may become negative. The effect has been analysed and the dependence of wave energy on the surface tension, Froude number (intensity of the background flow) and shear flow vorticity has been presented.

It has been shown that the potential energy remains positive regardless of the presence of a shear flow, whereas the kinetic energy depends on the parameters of a shear flow (e.g. velocity and vorticity). Namely, the kinetic energy may become negative, and consequently the total energy may become negative too.

It has also been shown that the virial theorem, i.e., the equality of period-averaged kinetic and potential energies [13], does not hold in shear flows. This agrees with the earlier findings for internal waves in two layer fluids [5,16,4], as well as for surface waves on a uniform flow [17] (the same conclusion has been made for surface waves on a linear shear flow [18], but the expression for the kinetic energy was incorrect).

Note also that in the Lecture Notes on linear wave theory [17] Dysthe has derived the formula for the density of wave energy, which is similar to our Eq. (27), but does not contain the term describing the effect of flow vorticity (here we call $E$ the wave energy just for the sake of brevity). Then, using the formula derived, Dysthe has shown that it can be presented in terms of conservation of the wave action (alias the density of number of quasi-particles in the quantum-mechanical terminology [4]), $N \equiv E_0/\omega_0 = E/(\omega_0 + \mathbf{U}\cdot\mathbf{k})$, where $E_0$ and $\omega_0$ pertain to waves on a stationary fluid, and $E$ and $\omega \equiv \omega_0 + \mathbf{U}\cdot\mathbf{k}$ pertain to waves in the coordinate frame moving with the velocity $-U$ with respect to the stationary fluid.

Such a simple relationship between the energy and frequency is possible only for a uniformly moving fluid. In our case, where the fluid flow is non-uniform in depth, the relationship between the energy and frequency is more complicated. Denoting the wave energy in the stationary fluid by $E_0^\pm = \rho(g+\gamma k^2)A^2/2$ (see [16,4, 17]), we can rewrite Eq. (27) as

$$E^\pm = \frac{E_0^\pm}{(g+\gamma k^2)|k|\tanh|k|h}\left(\omega^\pm - \mathbf{U}\cdot\mathbf{k}\right) \times \left[\omega^\pm + \left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)\frac{\alpha}{2}\tanh|k|h\right]. \quad (32)$$

The denominator on the right-hand side of Eq. (32) can be replaced by the expression which follows from the dispersion relation (9):

$$(g+\gamma k^2)|k|\tanh|k|h = (\omega^\pm - \mathbf{U}\cdot\mathbf{k})\left[\omega^\pm - (\mathbf{U}\cdot\mathbf{k})\left(1-\alpha\frac{\tanh|k|h}{U_0|k|}\right)\right].$$

After that Eq. (32) reads:

$$E^\pm = E_0^\pm \frac{\omega^\pm + \left(\hat{\mathbf{U}}_0\cdot\hat{\mathbf{k}}_0\right)(\alpha/2)\tanh|k|h}{\omega^\pm - (\mathbf{U}\cdot\mathbf{k})\left(1-\alpha\ \tanh(|k|h)/U_0|k|\right)}. \quad (33)$$

Introducing the notation

$$\omega_0^\pm = \omega^\pm - (\mathbf{U}\cdot\mathbf{k})\left(1-\alpha\frac{\tanh|k|h}{U_0|k|}\right), \quad (34)$$

we can finally rewrite Eq. (33) as

$$\frac{E^\pm}{\omega_0^\pm + (\mathbf{U}\cdot\mathbf{k})\ \left(1-\alpha\tanh(|k|h)/2U_0|k|\right)} = \frac{E_0^\pm}{\omega_0^\pm}. \quad (35)$$

For a uniformly moving fluid without shear ($\alpha = 0$) Eqs. (34) and (35) reduce to the well-known formula for the energy and Doppler-shifted frequency (see, e.g., [17]). In this case, the 'rest coordinate system' corresponds to the system which runs together with the fluid moving with the velocity $\mathbf{U}$. In this system surface waves propagate in the calm water and have a positive energy $E_0^\pm$ and frequency $\omega_0^\pm$.

For waves on a shear flow the 'rest coordinate system' runs with the velocity which depends on the wavenumber and vorticity $\alpha$: $(\mathbf{U}\cdot\mathbf{k})\left(1-\alpha\tanh|k|h/U_0|k|\right)$. Due to the combined effect of dispersion and shear it is impossible to transfer to such a coordinate system where all waves propagate on calm water.

Regarding physical interpretation of the condition (29), when NEWs appear in a shear flow, we should notice that, as has been shown in Section 3, in consistency with the linear approximation, the period-averaged wave energy can be presented as

$$\langle E\rangle = \frac{\rho}{2}\left\langle g\eta^2 + \frac{\sigma}{\rho}\left(\frac{\partial\eta}{\partial x}\right)^2 + \int_{-h}^{0}\left(u^2+v^2\right)dz\right\rangle + \rho\,U_0\,\langle\eta(x,t)u(x,0,t)\rangle\,. \quad (36)$$

From this expression it is clearly seen that the expression in the first angular brackets on the right-hand side represents essentially positive quantity, whereas the last term can be of either sign, i.e., positive or negative. Namely, thanks to this last term the total energy may become negative, if the perturbation of fluid surface $\eta(x,t)$ is in anti-phase with the perturbation of horizontal fluid speed $u(x,0,t)$.

In conclusion, we present an estimate of the conditions under which NEWs appear, both for a uniform flow and for a shear flow with a constant vorticity. To this end, let us choose the following parameters for the clean water of density $\rho = 10^3$ m$^3$, and surface tension 0.073 N/m at $T = 20$ °C. In this case, as follows from Eq. (31) with $\Omega = 0$, NEWs with $k_m = \sqrt{\rho g/\sigma} \approx 366.4$ m$^{-1}$ ($\lambda_m \approx 1.7$) cm appear on deep water at $U_0 = 23$ cm/s (Froude number $\mathrm{Fr} = \sqrt[4]{4S} = 0.74$) — this result is well-known [3,4]. The stronger the surface tension, the larger the wavelength $\lambda_m \sim \sqrt{\sigma}$. NEWs appear when the wave frequency at the local minimum reaches zero (see line $3^-$ in Fig. 2).

If the current profile varies linearly with the depth (see line 2 in Fig. 1), so that the vorticity is $\alpha/2 = U_0/2h$, then NEWs appear approximately at the same wavenumber (wavelength) and current speed at the water surface $U_0 = 23$ cm/s, but now the wave frequency at the local minimum is 26.9 s$^{-1}$, $T = 0.23$ s. From the value of the Froude number, $\mathrm{Fr} = 0.74$, we find the fluid depth $h = 0.75$ mm and the dimensionless wavenumber $\kappa = kh = 2.75$. With this value of $\kappa$ the deep-water approximation used here for the sake of simplicity is still satisfactorily applicable, because $\tanh 2.75 \approx 0.992$. Thus, the waves of smaller frequencies at higher current speeds can be unstable when the appropriate dissipation mechanism is taken into account. This may be of particular interest for the technological processes of evaporation and sputtering which deal with thin-film flows.

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## References

[1] M.E. McIntyre, On the 'wave momentum' myth, J. Fluid Mech. 106 (1981) 331–347.
[2] O. Bühler, Waves and Mean Flows, Cambridge University Press, Cambridge et al., 2009.
[3] Yu.A. Stepanyants, A.L. Fabrikant, Propagation of waves in hydrodynamic shear flows, Sov. Phys. Uspekhi 32 (9) (1989) 783–805.
[4] Yu.A. Stepanyants, A.L. Fabrikant, Propagation of Waves in Shear Flows, Nauka. Fizmatlit, Moscow, 1992 (in Russian) (Engl. transl., World Scientific, Singapore, 1998).
[5] R.A. Cairns, The role of negative energy waves in some instabilities of parallel flows, J. Fluid Mech. 92 (1979) 1–14.
[6] J. Pierce, Almost All About Waves, The MIT Press, Cambridge, Massachusetts and London, England, 1974.
[7] T.B. Benjamin, The threefold classification of unstable disturbances in flexible surfaces bounding inviscid flows, J. Fluid Mech. 16 (1963) 436–450.
[8] B.B. Kadomtsev, A.B. Mikhailovskii, A.V. Timofeev, Negative energy waves in dispersive media, Sov. Phys.—JETP 47 (1964) 2267–2269.
[9] A.B. Mikhailovskii, Theory of Plasma Instabilities, Vol. 1, second ed., Atomizdat, Moscow, 1975, vol. 2, 1977 (in Russian) (Engl. transl., 1st ed., Consultants Bureau, N. Y., 1974).
[10] M.V. Nezlin, Negative-energy waves and the anomalous Doppler effect, Sov. Phys. Uspekhi 19 (1976) 946–954.
[11] L.A. Ostrovsky, S.A. Rybak, L.Sh. Tsimring, Negative energy waves in hydrodynamics, Sov. Phys. Uspekhi 29 (11) (1986) 1040–1052.
[12] O. Phillips, The Dynamics of the Upper Ocean, Cambridge University Press, 1977.
[13] L.D. Landau, E.M. Lifshitz, Mechanics. Course of theoretical Physics, Vol. 1, fifth ed., Fizmatlit, Moscow, 2004 (in Russian) (Engl. transl., 3rd ed., Pergamon Press, Oxford, 2000).
[14] G.B. Whitham, Linear and Nonlinear Waves, Wiley Interscience Publ., John Wiley and Sons, N. Y., 1974.
[15] F.R. Gantmacher, Lectures in Analytical Mechanics, Nauka, Moscow, 1966 (in Russian) Engl. transl., Mir, Moscow, 1975).
[16] Yu.A. Stepanyants, Relationship between kinetic and potential energies in internal waves in the presence of shear currents, Izv., Acad. Sci. USSR, Atmos. Ocean. Phys. 21 (1985) 518–520.
[17] K. Dysthe, Lecture notes on linear wave theory. Lectures given at the summer school on: Water Waves and Ocean Currents, 21–29 June, 21 pp., 2004.
[18] S.A. Ellingsen, I. Brevik, How linear surface waves are affected by a current with constant vorticity, Eur. J. Phys. 35 (2014) 16.
[19] P.D. Thompson, The propagation of small surface disturbance through rotational flow, Ann. New York Acad. Sci. 5 (1949) 463–474.
[20] F. Biesel, Etude théorique de la houle en eau courante, Houille Blanche 5 (1950) 279–285.
[21] I.G. Jonsson, Wave–current interactions, in: B. Le Mehaute, D.M. Hanes (Eds.), The Sea, John Wiley, 1990, pp. 65–120.
[22] I. Kantardgi, Effect of depth current profile on wave parameters, Coast. Eng. 26 (3–4) (1995) 195–206.
[23] E. Wahlen, On rotational water waves with surface tension, Phil. Trans. R. Soc. A 365 (2007) 2215–2225.
[24] W. Choi, Nonlinear surface waves interacting with a linear shear current, Math. Comput. Simul. 80 (2009) 29–36.
[25] N.E. Kochin, I.A. Kibel, N.V. Roze, Theoretical Hydromechanics, sixth ed., Fizmatgiz, Moscow, 1963 (in Russian) (Engl. transl., John Wiley & Sons, Ltd., 1964).
[26] S. Tsao, Behaviour of surface waves on linearly varying currents, J. Geophys. Res. 79 (1957) 4498–4508.