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On quasi-static models hidden in Maxwell's equations [☆]

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ABSTRACT

In this paper we introduce the electromagnetic quasi-static models in a simple but meaningful way, relying on the dimensional analysis of Maxwell's equations. This analysis puts in evidence the three characteristic times of an electromagnetic phenomenon. It allows to define the range of validity of well-known models, such as the eddy-current (MQS) or the electroquasistatic (EQS) ones, and thus their pertinence to describe a given phenomenon. The role of the so-called "small parameters" of a model is explained in detail for two classical examples, namely a capacitor and a solenoid. We show how the MQS and EQS models result from having replaced fields by their first order truncations of Taylor expansions with respect to these small parameters. We finally investigate the connection between quasi-static models and circuit theory, clarifying the role of the fields with respect to classical circuit elements, and provide an example of application to study the electromagnetic fields in a simple case.

1. Introduction

Maxwell's equations (see for example [10]) are fundamental for the description of electromagnetic phenomena and valid over a wide range of spatial and temporal scales. The static limit of the theory is well defined. The electric and magnetic fields are given by the laws of Coulomb and Biot–Savart. As soon as there is any time dependence, we should in principle use the full set of Maxwell's equations with all their complexity. However, concrete problems in electromagnetism rarely require the solution of Maxwell's equations in full generality, because of various simplifications due to the smallness of some terms [4]. In this work, we try to quantify this smallness by means of a dimensional analysis [2,5] of Maxwell's equations. Indeed, some particular models in the low frequency limit, also known as quasi-static range (QS), emerge from neglecting particular couplings of electric and magnetic field related quantities. Following [12–14], we discuss the fact that there exists not one but indeed two dual Galilean limits called "electric" or EQS, and "magnetic" or MQS limits, the first including capacitive effects, the latter inductive effects. A dimensional analysis on the fields allows to emphasize the correct scaling yielding the two (limit) sets of Maxwell's equations. By means of detailed mathematical steps for two classical physical situations, we underline the role in the description of the fields' amplitude of the "small parameters" resulting from the dimensional analysis of Maxwell's equations. We provide simple numerical results on equivalent electric circuits to support the conclusions on the time-range validity of the considered quasi-static models. In some concrete applications however, at a certain frequency and for certain configurations of inductors, the separation between inductive and capacitive effects is not possible (see an example in [6]). In these cases, suitable formulations have then to be designed on the basis of the ones discussed in this work. The present work aims at proposing a new derivation of some existing quasi-static

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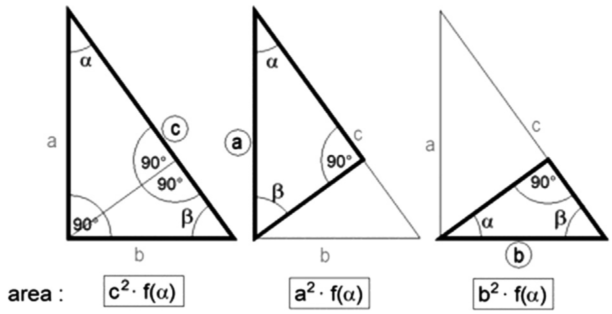

Fig. 1. The area of any triangle T depends on its size and shape which are defined by giving one edge, the largest c, and two angles α, β (for straight triangles, one angle, say α, is enough as $\beta = 90^\circ - \alpha$). Thus straight triangles with thick perimeter have area $c^2 f(\alpha)$ (left), $a^2 f(\alpha)$ (center), $b^2 f(\alpha)$ (right), respectively, with f a non-dimensional function of the angles which are dimensionless quantities too. Divide T into 2 non-overlapping smaller triangles T_1, T_2, by using the perpendicular to c as indicated, then $\text{area}(T) = \text{area}(T_1) + \text{area}(T_2)$ that yields $c^2 f(\alpha) = a^2 f(\alpha) + b^2 f(\alpha)$. Eliminating f, one gets $c^2 = a^2 + b^2$ without never specifying the form of f.

models by means of a dimensional analysis of Maxwell's equations. Moreover, we provide a condition stated on the basis of (physical) quantities related to the phenomenon, which allows to identify the mathematical model best suited for its investigation. To be more precise, the emphasis is put on the following points which cover both mathematical modeling and numerical validation. In Section 2 we recall the basis of dimensional analysis and we apply it in Section 3 to introduce the characteristic times for an electromagnetic phenomenon. In Section 4 we recall that the quasi-static models are Galilean limits of Maxwell's equations and underline that the field amplitude ratio matters in the selection of a limit. In Section 5, we scale Maxwell's equations using non-dimensional quantities naturally related to the previously introduced characteristic times and amplitude ratios, and state the model to be used according to the scaling. The mathematical part ends in Section 6 with a justification of the parameters introduced so far as the natural ones that appear when performing an asymptotic analysis of Maxwell's equations for two classical applications. Finally, in Section 7 we analyze the connection between quasi-static models and RCL circuits, and a numerical validation of the presented models is proposed in Section 8 together with some concluding remarks.

2. Dimensional analysis: known concepts

In physics and other sciences, we have to deal with distances or time intervals that, in a Galilean perspective, we are able to measure, comparing the first with a graduated meter and the second on a suitable clock. When we measure a quantity $\mathbf{g}$ with respect to a unit $\mathbf{u}$ we write it as $\mathbf{g} = g\mathbf{u}$ with g a real number. The surface of a (planar) square with side of size ℓ with respect to a fixed unit $\mathbf{u}$ is $\mathbf{a} = \ell^2\mathbf{u}^2$. The numbers g, ℓ, are however approximate due to errors in the measurement process, better if we express the measure of $\mathbf{g}$ or of $\mathbf{a}$ with respect to another quantity of the same kind chosen as unit. In this way, we introduce the dimension L of lengths and say that the surface has the dimension of the square of a length by writing

$$[\mathbf{a}] = [L]^2. \tag{1}$$

The unit of a physical quantity and its dimension are linked, but not identical concepts. The units of a physical quantity are defined by convention and related to some standard; *e.g.*, length may have units of meters, feet, inches, miles or micrometers; but any length always has the dimension of L, independent of what units are arbitrarily chosen to measure it. The concept of dimension is thus more abstract than that of unit: length is a dimension and meter is a unit $\mathbf{u}$ for lengths. If we change the system of units $\mathbf{u}$ for the length setting $\mathbf{u}' = \lambda\mathbf{u}$ then the surface in the new system of units becomes $\mathbf{a}' = \lambda^{-2}\mathbf{a}$ but still $[\mathbf{a}'] = [L]^2$. Similarly, for a volume we have $[L]^3$. In the case of angles θ, since their measure is expressed as ratio between lengths, we have $[\theta] = [L]^0$, thus angles have no dimension and the same for all trigonometric functions of angles. See Fig. 1 for a proof of the Pythagorean theorem in the Euclidean plane on the basis of these few concepts.

The main difficulty in dimensional analysis is the selection of the physical dimensions. First Isaac Newton (1686), who referred to it as the Great Principle of Similitude, then James Clerk Maxwell (1855) played a major role in establishing modern use of dimensional analysis by distinguishing mass M, length L, time T and current intensity I as fundamental quantities, while referring to others as derived (other quantities are also considered as fundamental but will not be involved in what follows). So, for example, by writing $[v] = [M]^0[L]^1[T]^{-1}[I]^0$ we say that speed has the dimension of a length divided by a time in any possible system of units. The selection of the fundamental dimensions – why the current I instead of the tension V which is easier to measure? – is a largely discussed subject in the literature and goes beyond the purpose of the present work.

Another important step is the idea that physical laws, such as force equals mass times acceleration, must be independent of the units used to measure the involved physical variables, here mass and acceleration. This led to the conclusion that meaningful laws must be homogeneous equations in their various systems of units, a result which was later formalized in the Vashy–Buckingham theorem [5]. Indeed, this theorem describes how every physically meaningful equation involving

Table 1
Parameter units in the MKSA system.

	μ	ϵ	σ	τ	ℓ
L	1	−3	−3	0	1
M	1	−1	−1	0	0
T	−2	4	3	1	0
I	−2	2	2	0	0

n variables can be equivalently rewritten as an equation of $n-m$ dimensionless parameters, where m is the number of fundamental dimensions used. Furthermore, it provides a method for computing these dimensionless parameters from the given variables, as we are going to show by an example in the electromagnetic context. Straightforward applications of this important idea are finding and checking relations among physical quantities by using their dimensions, simplifying a problem by reducing the number of physical parameters, checking the plausibility and coherence of derived models. In the present work, we rather focus on the latter.

3. Characteristic times of electromagnetic phenomena

We perform a dimensional analysis of the Maxwell's equations to put in evidence some characteristic quantities which allow to define the range of validity of a given model. We consider an electromagnetic phenomenon occurring in space–time sub-domain of $\mathbb{R}^3 \times \mathbb{R}^+$ of spatial characteristic length ℓ in a time duration τ. The spatial domain is a continuous medium with constitutive properties ϵ, μ, and σ, which are supposed to be constant for simplicity (otherwise they are time and space dependent tensors). Of course, it may be necessary to subdivide the original domain in sub-domains over which the electromagnetic parameters are not changing too much. Thus, the following analysis applies for each of such sub-domains.

Three characteristic times, namely τ_{em}, τ_e and τ_m, appear as soon as we represent τ and ℓ in terms of the fundamental physical parameters ϵ, μ and σ. In the MKSA system for example, expressed in terms of mass M (Kg), length L (m), time T (s) and current I (A), the parameters' dimensions are

$$[\mu]=[L][M][T]^{-2}[I]^{-2},$$
$$[\epsilon]=[L]^{-3}[M]^{-1}[T]^4[I]^2,$$
$$[\sigma]=[L]^{-3}[M]^{-1}[T]^3[I]^2.$$

Considering the numerical part of Table 1 as a 4×5 matrix, and remarking that the last two columns of the so-defined matrix contain just one non-zero unitary entry and that the last line is minus twice the second, the matrix rank is 3. Two parameters (τ and ℓ) can be expressed as functions of three others (μ, ϵ and σ). To this purpose, we seek for α, β, γ, c_1, c_2, and c_3 reals such that the following two ratios are dimensionless:

$$\tau/\big(\mu^\alpha\epsilon^\beta\sigma^\gamma\big)=O(1), \qquad \ell/\big(\mu^{c_1}\epsilon^{c_2}\sigma^{c_3}\big)=O(1).$$

The first ratio yields the following linear system

$$\begin{cases}\alpha-3\beta-3\gamma=0,\\ \alpha-\beta-\gamma=0,\\ -2\alpha+4\beta+3\gamma=1,\\ -2\alpha+2\beta+2\gamma=0\end{cases}$$

whose solution is $\alpha=0$, $\beta=1$, and $\gamma=-1$ (the fourth equation coincides with the second one up to a multiplicative factor -2). We introduce the first quantity $\tau_e=\epsilon/\sigma$ and we have $\tau/\tau_e=O(1)$. Indeed, τ_e is the electric charge diffusion time that is the characteristic time during which the simple electric charge decays in a conductor.

For the second ratio, we have to find c_1, c_2, and c_3 solution of a similar linear system with right-hand side equal to $(1,0,0,0)^t$. We thus get $c_1=-1/2$, $c_2=1/2$, and $c_3=-1$. We introduce $\ell^\star=(\sqrt{\epsilon/\mu})/\sigma$ and we have $\ell/\ell^\star=O(1)$. Since $\ell/\ell^\star=\mu\sigma\ell c=\mu\sigma\ell^2(c/\ell)$, a natural choice is to set $\tau_{em}=\ell/c$ and $\tau_m=\mu\sigma\ell^2$. The quantity τ_m is the current density diffusion time that is the characteristic time during which the current density (and hence the magnetic field) penetrates in a conductor. Its name is due to the fact that $D_m=1/(\mu\sigma)$ is the magnetic diffusion coefficient which has dimension $[L]^2[T]^{-1}$ and $\tau_m=\ell^2/D_m$. With these choices, $\tau_{em}^2=\tau_e\tau_m$, which is the time required for fields to propagate as an electromagnetic wave from one side to the other of Ω over a distance ℓ at the speed $c=1/\sqrt{\epsilon\mu}$. The origin of these waves is the coupling between the laws of Faraday and Ampère afforded by the magnetic induction and the displacement current. If either one or the other of these terms is neglected, so too is any electromagnetic wave effect. We note that τ and ℓ, thus the characteristic velocity $|v|=\ell/\tau$, are fixed by the problem features. Depending on the physical parameters μ, ϵ, and σ we specify the time intervals and thus suitable models from Maxwell's equations. In particular, in this work we are interested in the set of equations to be solved in the low frequency limit, that is when $|v|/c \ll 1$ and the difficulty is related to the fact that the quantity $|v|/c$ is not the only indicator that matters, as we are going to explain in the next section.

4. Quasi-statics with moving media

From now on, quantities with "primes" are related to the moving reference system $\mathcal{R}'$ and those without are related to the fixed reference system $\mathcal{R}$. Assume that $\mathcal{R}$ and $\mathcal{R}'$ have the same origin at $t = t' = 0$: the coordinate systems $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$ are said to be in standard configuration. A Lorentz transformation between $\mathcal{R}$ and $\mathcal{R}'$ in standard configuration acts on space–time coordinates as follows [8]

$$\mathbf{x}' = \gamma(\mathbf{x} - \mathbf{v}t),$$
$$t' = \gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right), \quad \gamma = 1\Big/\sqrt{1 - \frac{|v|^2}{c^2}}, \tag{2}$$

where $\mathbf{v}$ is the relative velocity between $\mathcal{R}'$ and $\mathcal{R}$ and $|v|$ its modulus. When $|v| \ll c$ (that yields $\gamma \sim 1$), under the validity of the causality principle $\Delta x = |v|\Delta t \ll c\Delta t$, transformations (2) reduce to Galilean ones

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \qquad t' = t. \tag{3}$$

For the fields, as given by Einstein and Laub in 1908 [7], we have

$$E' = \gamma(E + \mathbf{v} \times B) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot E)}{|v|^2},$$
$$B' = \gamma\left(B - \frac{\mathbf{v}\times E}{c^2}\right) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot B)}{|v|^2},$$
$$D' = \gamma\left(D + \frac{\mathbf{v}\times H}{c^2}\right) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot D)}{|v|^2},$$
$$H' = \gamma(H - \mathbf{v}\times D) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot H)}{|v|^2}. \tag{4}$$

To take the limit for $|v| \ll c$ is not only sufficient to set $\gamma \sim 1$ in (4). Indeed, one would obtain for example $E' = E + v \times B$ and $B' = B - (v \times E)/c^2$ which do not respect the group composition property. Note that the group composition property is a key point to understand the validity of a physical transformation and is the mathematical expression of the relativity principle. Starting from (4) with $\gamma \sim 1$, if $e \ll cb$, the term $B - (\mathbf{v}/c) \times (E/c)$ gives $b - (|v|/c)(e/c) \sim b$, thus we get

$$E' = E + \mathbf{v}\times B, \qquad B' = B,$$
$$D' = D + (\mathbf{v}\times H)/c^2, \qquad H' = H,$$
$$\rho' = \rho - \mathbf{v}\cdot J/c^2, \qquad J' = J. \tag{5}$$

On the other hand, if $e \gg cb$, the two terms $(|v|/c)$ and (e/c) equilibrate each other and have to be kept, whereas $e + |v|b \sim e$ and thus we have

$$E' = E, \qquad B' = B - (\mathbf{v}\times E)/c^2,$$
$$D' = D, \qquad H' = H - \mathbf{v}\times D,$$
$$\rho' = \rho, \qquad J' = J - \rho\mathbf{v}. \tag{6}$$

Similar transformations for potentials in the two limits and other details can be found in [14]. Note that in the Galilean regime, we make the assumption that the force F and the charge q are invariant when going from $\mathcal{R}$ to $\mathcal{R}'$, *i.e.*, $F' = F$ and $q' = q$. The Lorentz force $F' = q'E'$ gives $F = q(E + \mathbf{v}\times B)$ in the magnetic limit and $F = qE$ in the electric.

Constitutive relations as well depend on the considered Galilean limit (this fact was overlooked in [12]). We recall that in the moving reference $\mathcal{R}'$, the constitutive relation between B and H reads $B' = \mu H'$. When reported all quantities to $\mathcal{R}$, in the magnetic limit, one has

$$H = B/\mu, \qquad D \simeq \epsilon E + \left(\epsilon - \frac{1}{\mu c^2}\right) v \times B,$$

but in the electric limit one should rather use

$$H \simeq B/\mu - \left(\epsilon - \frac{1}{\mu c^2}\right)\frac{v\times E}{c^2}, \qquad D = \epsilon E.$$

One can be bothered by the presence of c in the above expressions despite we are considering phenomena where $|v| \ll c$. In quasi-static electromagnetism, the appearing velocity is $c_u = 1/\sqrt{\epsilon_0\mu_0}$. This velocity is independent of specific units (same value with Gaussian or SI units) and arises from using only action-at-a-distance forces in which an instantaneous propagation is assumed [9]. It can thus be considered as a fundamental constant in nature. We are used to identify c_u with c the speed of light in vacuum because these velocities have the same numerical value. We have to remember that the speeds c_u and c emerge from different physical considerations (Maxwell indeed has been the first one who stated $c_u = c$ in 1862) but we are not going to develop this point here. Note that the Galilean electrodynamics of moving media is still open to research as many aspects have not been completely understood yet.

5. Limit models of Maxwell's equations

Let us set the electric field $E = e\mathcal{E}$ and $B = b\mathcal{B}$, where e, b are reference quantities (also called electric and magnetic fields scaling factors) whereas $\mathcal{E}$, $\mathcal{B}$ are non-dimensional quantities of order 1. We just recall that in dimensional analysis the spatial (resp. time) differentiation $\partial_x E$ (resp. $\partial_t E$) is equivalent to $\frac{e}{\ell}\partial_{x'}\mathcal{E}$ (resp. $\frac{e}{\tau}\partial_{t'}\mathcal{E}$) where $x = \ell x'$ (resp. $t = \tau t'$). Moreover, we adopt the notation $a \sim b$ to say that the quantities a and b have the same magnitude order, whereas $a \simeq b$ when a and b are approximately equal.

The data of an electromagnetic problem are initial and boundary conditions for the fields E and B together with the sources, namely the charge density ρ and the current density J which are linked by the continuity equation

$$\partial_t \rho + \nabla \cdot J = 0. \tag{7}$$

Maxwell's equations describing the electromagnetic phenomenon read

$$\begin{aligned}
&\nabla \times E = -\partial_t B,\\
&\nabla \times H = \partial_t D + J,\\
&\nabla \cdot D = \rho,\\
&\nabla \cdot B = 0,
\end{aligned} \tag{8}$$

where $D = \epsilon E$, $B = \mu H$ and $J = \sigma E$ are the constitutive relations.

We now perform a scaling of the equations based on the characteristic times and amplitude ratios introduced in Sections 3 and 4. A rather similar analysis was firstly presented in [13]. We thus obtain from Faraday's law

$$\nabla' \times \mathcal{E} = -\frac{\ell}{\tau}\frac{b}{e}\partial_{t'}\mathcal{B}. \tag{9}$$

Thus, the first scaling appears, that is

$$e \sim |v| b. \tag{10}$$

Eq. (9) can also be written as

$$\nabla' \times \mathcal{E} = -\frac{\tau_{em}}{\tau}\frac{cb}{e}\partial_{t'}\mathcal{B}. \tag{11}$$

Using similar arguments, from Ampère's law we get

$$\nabla' \times \mathcal{H} = \frac{\ell}{\tau}\frac{d}{h}\partial_{t'}\mathcal{D} + \frac{\ell j}{h}\mathcal{J}$$

which becomes, using the constitutive relations,

$$\nabla' \times \mathcal{B} = \frac{\ell}{\tau}\frac{\epsilon\mu e}{b}\partial_{t'}\mathcal{E} + \frac{\ell\mu\sigma e}{b}\mathcal{E}. \tag{12}$$

Thus the second scaling appears (assume $\mathcal{J} = 0$ for a moment), that is

$$b \sim \frac{|v|}{c^2}e. \tag{13}$$

Eq. (12) can also be written as either

$$\nabla' \times \mathcal{B} = \frac{\tau_{em}}{\tau}\frac{e}{cb}\partial_{t'}\mathcal{E} + \frac{\tau_m}{\tau_{em}}\frac{e}{cb}\mathcal{E}, \tag{14}$$

or

$$\nabla' \times \mathcal{B} = \frac{\tau_{em}}{\tau}\frac{e}{cb}\partial_{t'}\mathcal{E} + \frac{\tau_{em}}{\tau_e}\frac{e}{cb}\mathcal{E}. \tag{15}$$

When $|v| \sim c$, the two scalings (10) and (13) are the same. Suppose now that $|v| \ll c$, the two scalings are different and if one replaces the expression of b given in (13) into the expression of e given in (10), we gets $|v| \sim c$ which is in

Table 2
Range of the characteristic time τ w.r.t. τ_{em}.

Full set of Maxwell's eq.	Quasi-static regime	Static regime
$0 \leqslant \tau \leqslant \tau_{em}$	$\tau_{em} \leqslant \tau \leqslant \tau_m, \tau_e$	$\tau \geqslant \tau_m, \tau_e$

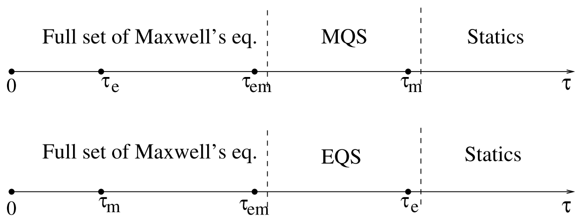

Fig. 2. Case (i) down, and case (ii) up.

contradiction with the starting assumption $|v| \ll c$. This means that when $|v| \ll c$, the two scalings are not simultaneously valid, thus the Faraday's and generalized Ampère's laws cannot be coupled in certain regimes, and this is what we are going to detail right below.

Note that comparing $|v|$ to c is the same as comparing τ_{em} to τ, since we have $|v|/c = \tau_{em}/\tau$. Whether we ignore the magnetic induction and use the EQS approximation, or neglect the displacement current and make a MQS approximation, times of interest τ must be long compared to the time τ_{em} required for an electromagnetic wave to propagate at the velocity c over the characteristic (largest) length ℓ of the system (see Table 2). Thus, an electromagnetic phenomenon is considered to happen in the low frequency range if $\tau_{em}/\tau < 1$ (generally, $\tau_{em}/\tau < 0.1$ is enough). But looking at Eq. (11) or (14), we remark that the term τ_{em}/τ is multiplied by the quantity $e/(cb)$ that could in principle be much larger than 1 and other terms such as τ_m/τ_{em}, τ_{em}/τ_e have to be taken into account. Table 2 summarizes the different possibilities. Suppose for example that $\tau_{em}/\tau \ll 1$ (generally, $\tau_{em}/\tau < (0.1)^2$ is enough and corresponds to the static regime), then, if $cb/e \approx \tau/\tau_{em}$ (thus $cb \gg e$), Eqs. (11) and (14) describe the current flow in perfect conductors whereas, if $cb/e \approx \tau_{em}/\tau$ (thus $cb \ll e$), the same equations allow to describe the electric field in perfect insulators. The more realistic situation where no perfect materials are present can be described by Eqs. (11) and (14) with $cb/e \approx \tau_m/\tau_{em} = \tau_{em}/\tau_e$. In this latter case, cb/e is not strictly related to τ. Three different situations to face:

(i) $\tau_m \ll \tau_{em} \ll \tau_e$ and $\tau_{em} < \tau < \tau_e$,
(ii) $\tau_e \ll \tau_{em} \ll \tau_m$ and $\tau_{em} < \tau < \tau_m$,
(iii) $\tau_m \approx \tau_{em} \approx \tau_e$ and $\tau_{em} \ll \tau$.

For a given characteristic time τ, it is clear from Fig. 2 that the region described by the quasi-static laws is limited in size. Systems can often be divided into subregions that are small enough to be quasi-static but, by virtue of being linked through their boundaries, are dynamic in their behavior. With the elements regarded as the subregions, electric circuits are an example, that we will treat numerically later on. In the physical world of perfect conductors and free space that we consider here, it is the geometry of the conductors that determines whether these subregions are EQS or MQS.

5.1. Case (i)

Here we have that $cb \ll e$ and Eq. (11) gives

$$\nabla' \times \mathcal{E} \simeq \mathbf{0}.$$

Note that the condition $cb \ll e$ is compatible with the scaling (13). Indeed, $cb \ll e$ with (10) would result in $|v|b \gg cb$ thus $|v| \gg c$ which is in contradiction with the assumption $|v| \ll c$.

In case of high frequencies, that is when $\tau \approx \tau_e$, also the first term in the r.h.s. of (15) is of order 1 and the equations to be solved are

$$\nabla \times E \simeq 0, \qquad \nabla \times H = J + \partial_t D,$$
$$\nabla \cdot D = \rho, \qquad \partial_t \rho + \nabla \cdot J = 0, \tag{16}$$

called the ElectroQuasiStatic equations (EQS). According to the authors' knowledge, the EQS model has not been yet analyzed from the mathematical point of view.

In the case of extremely low frequencies, that is when $\tau_e \ll \tau$, since

$$\frac{e}{cb}\frac{\tau_{em}}{\tau} \approx \frac{\tau_e}{\tau},$$

the first term in the r.h.s. of (15) is negligible w.r.t. the second term and the equations to be solved are

$$\nabla \times E \simeq 0, \qquad \nabla \times H \simeq J,$$
$$\nabla \cdot D = \rho, \qquad \nabla \cdot B = 0, \tag{17}$$

called the Quasi-Stationary Conduction (QSC) equations.

In case of low frequencies, that is when $\tau_e < \tau$, depending on whether we neglect or not the first term in the r.h.s. of (15) w.r.t. the second term, the equations to be solved are the QSC or the EQS ones, respectively.

5.2. Case (ii)

Here, we have $cb \gg e$ and Eq. (15) gives

$$\nabla' \times \mathcal{H} \simeq \frac{\tau_{em}}{\tau_e}\frac{e}{cb}\mathcal{J}$$

as the term with the displacement current is negligible. Now, the condition $cb \gg e$ is compatible with the scaling (10). Indeed, $cb \gg e$ with (13) would yield $|v| \gg c$ which is in contradiction with the assumption $|v| \ll c$.

In case of high frequencies, that is when $\tau \approx \tau_m$, the r.h.s. of (11) is of order 1 and the equations to be solved are

$$\nabla \times E = -\partial_t B, \qquad \nabla \times H \simeq J,$$
$$\nabla \cdot B = 0, \qquad \nabla \cdot J = 0, \tag{18}$$

called the MagnetoQuasiStatic (MQS) or eddy-current equations. The MQS model has been analyzed from the mathematical point of view in [1].

In case of extremely low frequencies, that is when $\tau_m \ll \tau$, since

$$\frac{cb}{e}\frac{\tau_{em}}{\tau} \approx \frac{\tau_m}{\tau},$$

Eq. (11) gives $\nabla' \times \mathcal{E} \simeq \mathbf{0}$ and the QSC equations have to be solved.

In case of low frequencies, that is when $\tau_m < \tau$, depending on whether the r.h.s. of (11) is negligible or not, the equations to be solved are the QSC or the MQS ones, respectively.

5.3. Case (iii)

Here we have $cb \approx e$ and $\tau_m/\tau \approx \tau_{em}/\tau \approx \tau_e/\tau$ thus the r.h.s. of (11) and the first term in the r.h.s. of (14) are of the same order of magnitude. When $cb \approx e$ the two scalings (10) and (13) coincide.

In case of extremely low frequencies, that is when $\tau_m \ll \tau$, the situation is that described by the QSC equations.

In case of low frequencies, that is when $\tau_m < \tau$, we can have the QSC situation if we decide to neglect both the r.h.s. of (11) and the first term in the r.h.s. of (14). Otherwise, the equations to be solved are the full Maxwell's equations (8) with (7) but in the case $\tau_{em} < \tau$ (the propagation may still be negligible). This latter case is referred to as the ElectroMagneticQuasiStatic (EMQS) model. Darwin model described in [11,15,16] is a variant of case (iii) in the low frequency limit where we keep only the Coulomb part E_C of E in the displacement current, but will be not discussed here (see [15,16] for the derivation and analysis of the Darwin model, and [11] for physical considerations on this model).

5.4. Visualization

In order to underline the dependence of τ_m and τ_e on the length ℓ, it is better to consider a two-dimensional visualization, where one axis reflects the effect of τ and the other that of ℓ, as firstly proposed in [3]. We thus consider the plane (x, y) where $x := \log(\tau/\tau_{em})$ and $y := \log(\ell/\ell^*)$ and we separate it in sectors by remarking that

$$\tau = \tau_{em}, \qquad \log(\tau/\tau_{em}) = \log(1) \quad (x = 0),$$
$$\tau = \tau_e, \qquad \log(\tau/\tau_{em}) = \log(\ell^*/\ell) \quad (y = -x),$$
$$\tau = \tau_m, \qquad \log(\tau/\tau_{em}) = \log(\ell/\ell^*) \quad (y = x).$$

The quasi-static regime, characterized by $\tau_{em} < \tau$, is located where $x > 0$ on the plane (x, y) of Fig. 3.

6. The role of the small parameter

In the quasi-static regime, thank to a dimensional analysis of Maxwell's equations, we have seen that a "small parameter", say $\alpha = |v|/c$, has be compared with others, such as $e/(cb)$. We now develop the mathematical details for two classical cases (capacitor and solenoid) to show that, on the one hand, α is the natural parameter to express field amplitudes as Taylor expansions of the form $u_\alpha = u_0 + \alpha u_1 + \alpha^2 u_2 + \cdots$ and, on the other hand, the quasi-static models such as EQS or MQS correspond to a truncation of the Taylor expansion of the fields' amplitude to 1st order in α.

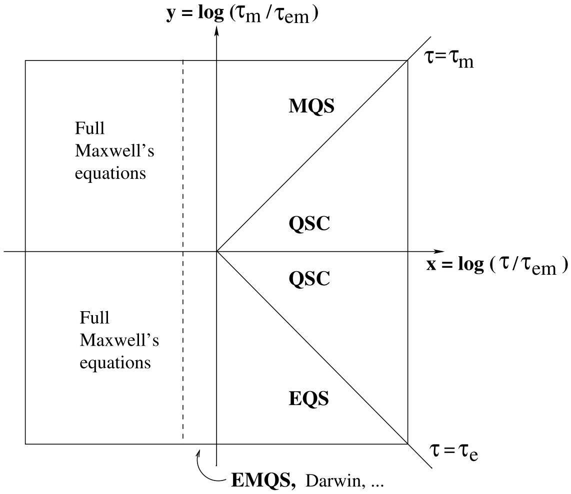

Fig. 3. Graphical representation of electromagnetic model validity.

6.1. Capacitor with flat circular electrodes at a forced sinusoidal regime

Let us consider a capacitor with two co-axial circular flat discs as electrodes at a distance d and finite radius $a \gg d$. The capacitor is submitted to a forced sinusoidal regime and electric charges appear on the two discs. The electric field belongs to the plane $\mathcal{P}$ of symmetry of the charge distribution over the capacitor plates. We adopt cylindrical coordinates (r, θ, z) in such a way that the z-axis coincides with the axis of the parallel plates and that the plane $z = 0$ is in the gap at distance $d/2$ from both plates. We compute the field amplitude twice, by solving Maxwell's equations and by relying on the perturbation approach. In this configuration, E and B are independent of the rotation angle θ. For E, we have $E(r,z,t) = e_r(r,z,t)\mathbf{i}_r + e_z(r,z,t)\mathbf{i}_z$ and, for B, we can write $B(r,z,t) = b_\theta(r,z,t)\mathbf{i}_\theta$, since $\nabla \times E = \epsilon_0 \partial_t E$. We note that

$$\nabla \times E = \frac{1}{r}(\partial_\theta e_z - \partial_z e_\theta)\mathbf{i}_r + (\partial_z e_r - \partial_r e_z)\mathbf{i}_\theta + \frac{1}{r}\big(\partial_r(re_\theta) - \partial_\theta e_r\big)\mathbf{i}_z = b_\theta \mathbf{i}_\theta$$

with b_θ very small as $e_r/e_z \approx d/a \ll 1$. In the gap, $\rho = 0$ thus $\nabla \cdot E = 0$ which in cylindrical coordinates gives

$$\nabla \cdot E = \frac{1}{r}\partial_r(re_r) + \frac{1}{r}\partial_\theta e_\theta + \partial_z e_z = 0.$$

This yields e_z constant function in z, since $e_r \ll e_z$, and thus $E(r,t) = e_z(r,t)\mathbf{i}_z$. As a consequence $B(r,t) = b_\theta(r,t)\mathbf{i}_\theta$. Coupling the two equations $\nabla \times E = -\partial_t B$ and $\nabla \times B = c^{-2}\partial_t E$, the component e_z of E verifies the equation

$$\partial_{rr} e_z + \frac{1}{r}\partial_r e_z - \frac{1}{c^2}\partial_{tt} e_z = 0. \tag{19}$$

Assuming a sinusoidal regime of frequency ω,

$$E(r,t) = E_0 A(r) e^{i\omega t}\mathbf{i}_z, \qquad B(r,t) = \frac{-iE_0}{\omega} A'(r) e^{i\omega t}\mathbf{i}_\theta \quad \left(A'(r) = \frac{dA(r)}{dr}\right).$$

Eq. (19) can be written as $A''(r) + \frac{1}{r}A'(r) + \frac{\omega^2}{c^2}A(r) = 0$. Let us make the change of variable $u = \omega r/c$ (note that $u \approx \alpha$), knowing that $d_r A(u) = d_r u\, d_u A$ and that $d_{rr} A = (\omega/c)^2 d_{uu} A$, Eq. (19) becomes

$$A''(u) + \frac{1}{u}A'(u) + A(u) = 0 \tag{20}$$

whose solution has the form $A(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}(\frac{u}{2})^{2k}$. We now obtain the same result differently. Let us set $E_0(r,t) = e_z(r,t)\mathbf{i}_z = e_0(r)e^{i\omega t}\mathbf{i}_z$. Applying alternatively Ampère's and Faraday's laws, we get

$$E_0(r,t) \to \text{Ampère} \to B_1(r,t) = e_0 \frac{i\omega r}{2c^2} e^{i\omega t}\mathbf{i}_\theta,$$

$$B_1(r,t) \to \text{Faraday} \to E_2(r,t) = -e_0 \frac{\omega^2 r^2}{4c^2} e^{i\omega t}\mathbf{i}_z,$$

$$E_2(r,t) \to \text{Ampère} \to B_3(r,t) = -e_0 \frac{i\omega^3 r^3}{16c^4} e^{i\omega t}\mathbf{i}_\theta,$$

$$B_3(r,t) \to \text{Faraday} \to E_4(r,t) = e_0 \frac{\omega^4 r^4}{64c^4} e^{i\omega t}\mathbf{i}_z$$

and so on. Thus $E(r,t) = e_0[1 - \frac{1}{4}(\frac{\omega r}{c})^2 + \frac{1}{64}(\frac{\omega r}{c})^4 \cdots]e^{i\omega t}\mathbf{i}_z$, which looks like $E(r,t) = e_0[1 - \frac{1}{4}u^2 + \frac{1}{64}u^4 \cdots]e^{i\omega t}\mathbf{i}_z = e_0 A(u)e^{i\omega t}\mathbf{i}_z$, and $B(r,t) = \frac{ie_0}{c}[\frac{1}{2}\frac{\omega r}{c} - \frac{1}{16}(\frac{\omega r}{c})^3 \cdots]e^{i\omega t}\mathbf{i}_\theta$. At this point, if we truncate the series of $E(r,t)$ and $B(r,t)$ at the 1st order in u, we get $|E/(cB)| \approx c/(\omega r) \approx \tau/\tau_{em}$ as obtained in the previous section for the EQS approximation.

6.2. *Infinite solenoid at a forced sinusoidal regime*

Let us consider a solenoid of height h, with circular windings of radius $a \ll h$ crossed by a sinusoidal current I of frequency ω. The magnetic field is orthogonal to the plane $\mathcal{P}$ of symmetry of the current distribution in the windings of the solenoid. We adopt cylindrical coordinates (r,θ,z) in such a way that the z-axis coincides with the axis of the solenoid and that the plane $z = 0$ is at distance $h/2$ from both solenoid extremities. We compute the field amplitude twice, by solving Maxwell's equations and by relying on the perturbation approach. In this configuration, E and B are independent of the rotation angle θ. For B, we can write $B(r,t) = b_z(r,t)\mathbf{i}_z$ since $a \ll h$ and, for E, we have $E(r,t) = e_\theta(r,t)\mathbf{i}_\theta$ since $\nabla \times E = -\partial_t B$. Coupling the two equations $\nabla \times E = -\partial_t B$ and $\nabla \times B = c^{-2}\partial_t E$, the component b_z of B verifies the equation

$$\partial_{rr} b_z + \frac{1}{r}\partial_r b_z - \frac{1}{c^2}\partial_{tt} b_z = 0. \tag{21}$$

Assuming a sinusoidal regime of frequency ω, we can write $B(r,t) = b_0 A(u)e^{i\omega t}$ with $A(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}(\frac{u}{2})^{2k}$ and $u = \omega r/c$. We now go through the perturbation approach, applying once again Ampère and Faraday laws alternatively. Imposing $(\nabla \times B)_\theta \mathbf{i}_\theta = c^{-2}\partial_t e_\theta \mathbf{i}_\theta$ yields $\partial_r b_z = c^{-2}\partial_t e_\theta$ where

$$\partial_t e_\theta = -c^2 \partial_r u A'(u) b_0 e^{i\omega t} = -c\omega b_0 A'(u)e^{i\omega t}.$$

We thus get

$$E(r,t) = -\frac{c\omega b_0}{i\omega} A'(u)e^{i\omega t}\mathbf{i}_\theta = icb_0 A'(u)e^{i\omega t}\mathbf{i}_\theta.$$

Recalling the expression of $A'(u)$, we obtain

$$E(r,t) = icb_0\left[\frac{1}{2}\frac{\omega r}{c} - \frac{1}{16}\left(\frac{\omega r}{c}\right)^3 \cdots\right]e^{i\omega t}\mathbf{i}_\theta.$$

At this point, if we truncate the series of $E(r,t)$ and $B(r,t)$ at the 1st order in u, we get $|E/(cB)| \approx (\omega r)/c \approx \tau_{em}/\tau$ as obtained in the previous section for the MQS approximation.

To summarize the mathematical steps presented above, in the quasi-static regimes where $u = \omega r/c \ll 1$ (e.g., if one takes $r = 1$ m, we have $\omega \ll c/r = 310^8$ Hz with $c = 310^8$ m/s, and thus $f = \omega/(2\pi) \ll 50$ MHz) the Taylor expansion of the fields' amplitude truncated at the 1st order in u, we have

$$\text{capacitor:}\quad E(r,t) = e_0(r)e^{i\omega t}\mathbf{i}_z, \qquad B(r,t) = \frac{ie_0(r)}{c}\frac{\omega r}{2c}e^{i\omega t}\mathbf{i}_\theta,$$

$$\text{solenoid:}\quad B(r,t) = b_0(r)e^{i\omega t}\mathbf{i}_z, \qquad E(r,t) = icb_0(r)\frac{\omega r}{2c}e^{i\omega t}\mathbf{i}_\theta. \tag{22}$$

Therefore, in the capacitor the electric field prevails since $|E/(cB)| \approx c/(\omega r) \approx \tau/\tau_{em} \gg 1$, whereas in the solenoid the magnetic field dominates as $|E/(cB)| \approx (\omega r)/c \approx \tau_{em}/\tau \ll 1$, in agreement with the validity ranges of the EQS and MQS models, respectively.

7. Quasi-static systems as *RCL* circuits and a numerical example

In [13], the authors develop a connection between the quasi-static models and circuit theory. In this section, we try to analyze deeply this connection and solve very simple classical numerical examples taking inspiration from [13]. It is well known that the EQS model, due to the presence of the displacement current, describes capacitance effects whereas the MQS one, due to the Ampère law, includes inductive effects. To understand this fact we try to clarify the role of the actors starring in Maxwell's equations such as B, D, J, ... with respect to the classical circuit elements such as the resistance R, the capacity C and the inductance L.

Fig. 4. Porous (left) versus elastic (right) medium. The total discharge Q of a fluid through a porous medium is proportional to the pressure drop $P_2 - P_1$ (Darcy's law). The deformation $\bar{x}$ of an elastic membrane or spring is proportional to the applied force $P_2 - P_1$ (Hooke's law).

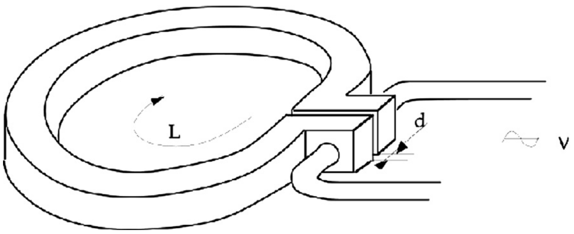

Fig. 5. A case where capacitive effects are not negligible: when $d \ll L$ (here $\ell = L$) the electric field between the plates of distance d is intense and the displacement current $\epsilon_0 \partial_t E$ behaves as an additional source of current (courtesy of A. Bossavit [4]).

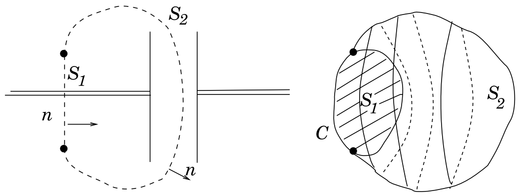

Fig. 6. A two-dimensional section of an electric circuit with an Ohmic conductor connected to two parallel electrodes. The two black points denote the intersection between a circular line $\mathcal{C}$ going around the conductor out of the electrodes and the plane of the picture. The dashed line is the intersection of two surfaces, ending in $\mathcal{C}$, with the plane of the picture. Surface S_2 (resp. S_1) does (resp. does not) contain an electrode.

7.1. The role of R, C and L

Each time there is an energy loss in the considered system, we can compare it to an electric circuit with a resistance R. The current J flowing in an Ohmic conductor is proportional to the voltage drop at the extremities of the conductor and causes energy losses due to Joule's effect. We can thus assimilate an Ohmic conductor to an electric circuit characterized by a resistance $R = \ell/(S\sigma)$, with ℓ the length, S the cross-section, σ the electric conductivity of the conductor. It is interesting to remark that the electric charges move in an Ohmic conductor as a viscous fluid in a porous medium (see Fig. 4 (left)) and the difference of voltage $\bar{V}$ at the extremities of the conductor has the same role as the difference of pressure $P_2 - P_1$ which exists at the extremities of the porous medium. The mechanical counterpart of the term containing R in the *RCL* circuit equation is the friction term $f\dot{x}$ proportional to the velocity of the displacement x.

Each time there is an energy accumulation in the system, we can assume the presence of a capacity C in the equivalent electric circuit. A capacitor works as a spring or an elastic membrane (see Fig. 4 (right)) with elastic coefficient κ. The mechanical counterpart of the term depending on C in the *RCL* circuit equation is κx proportional to the displacement x. The Hooke's law states that the deformation of the spring or membrane is $\bar{x} = (1/\kappa)F$ where $F = (P_2 - P_1)S$ is the applied force, where S is the cross-section of the spring or membrane. When $F = 0$, the membrane is at rest ($\bar{x} = 0$), and when $F \neq 0$, the membrane is deformed thus stocking energy. We remark that the accumulated energy is conserved until the membrane or spring is not put again in the rest position. Maxwell introduces an electric displacement D for the charges on the plates of the capacitor which is proportional to the applied force E $(= -\nabla V)$ through an electric elastic coefficient ϵ of the material between the plates. Analogously to the Hooke's law, we have $D = \epsilon E$. A similar behavior occurs in dielectric media, which in absence of an exterior electric field are not polarized and as soon as an exterior electric field is applied, they present a polarization $P \neq 0$. If $\bar{x} = (1/\kappa)F$ is the displacement, then $\partial_t \bar{x} = (1/\kappa)\partial_t F$ is the displacement velocity. Similarly, $\partial_t D = \epsilon \partial_t E$ gives the famous displacement current added by Maxwell to the Ohmic current in Ampère's law in order to take into account the capacitive effects of the capacitor.

As explained in [4], for the conductor presented in Fig. 5, the capacitance effects due to the presence of the gap cannot be neglected. We recall that the difference of voltage is, in the capacitor, $\bar{V} \sim ed$ and, in the conductor, $\bar{V} \sim \ell j/\sigma$. The ratio between $\epsilon e/\tau$, the current in the gap, and j the current in the conductor, is of order $\epsilon/(\tau\sigma)(\ell/d)$, and thus cannot be negligible if ℓ/d gets large with $\epsilon/(\tau\sigma) \approx 1$. The capacitance of the gap ($C = \epsilon/d$) cannot be neglected in the computation when its product with the resistance ($R = \ell/\sigma$) is $\approx \tau$.

On the simple example shown in Fig. 6, we understand why the displacement current allows to close the circuit in presence of electrodes. In the low frequency regime, out of the electrodes, Ampère's law holds, the electric field is negligible and $|\partial_t D| \ll |J|$. Thus $\nabla \times H = J$. In the gap, we have that $|J| = 0$ and thus we cannot have $|\partial_t D| \ll |J|$. Indeed, $\nabla \times H = \partial_t D$.

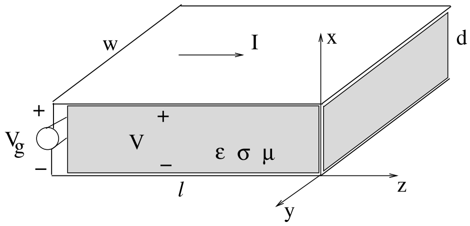

Fig. 7. A linear material characterized by constant values of σ, ϵ, μ, fills in the space between two perfectly conducting plane electrodes of dimensions $\ell \times w$ and distance d. The plates are connected on the left to a voltage generator and are open on the right.

We know that the circulation of H on a closed line around an electric circuit equals the current I flowing in the circuit. For the Stokes theorem we have

$$I = \oint_{\mathcal{C}} H \cdot \mathbf{t} = \int_{S_1(\mathcal{C})} \nabla \times H \cdot \mathbf{n}_1 = \int_{S_2(\mathcal{C})} \nabla \times H \cdot \mathbf{n}_2.$$

Indeed,

$$\int_{S_1(\mathcal{C})} \nabla \times H \cdot \mathbf{n}_1 = \int_{S_1(\mathcal{C})} J \cdot \mathbf{n}_1 = I,$$

and

$$\int_{S_2(\mathcal{C})} \nabla \times H \cdot \mathbf{n}_2 = \frac{d}{dt} \int_{S_2(\mathcal{C})} D \cdot \mathbf{n}_2 = I.$$

Two parallel electrodes are equivalent to an insulator between two conductors. If we modify the voltage between the electrodes, the electric field changes and the electric charges start to move in the conductors without producing a conducting current in a circuit since the circuit is open at the electrodes. There is a local motion of charges which is called displacement current related to the polarization of the material filling the gap between the electrodes.

Finally, each time there is a kinetic energy in the system, we can assume the presence of an inductance L in the equivalent electric circuit. The inductance works differently from a capacitor or a resistance. Once the inductance has been "charged" in electromagnetic inertia, neither it dissipates the energy as the resistance would do nor it can stock its energy as a capacitor. Let us compare the expression $\mathcal{L}_m = \frac{1}{2} m|v|^2$ of the kinetic energy of a mass m at speed $|v|$ to that $\mathcal{L}_e = \frac{1}{2} L I^2$ of the electromagnetic energy associated to the inductance L in a circuit with a circulating current I. In a mechanical analogy, we can say that an inductance behaves like an inertial mass. A mass with a kinetic energy is able to move for a moment even if there is no external force acting on it. The same occurs to the current in an electric circuit when the voltage at the extremities of the circuit drops down. The current still circulates for a moment in the circuit because of the inductance giving back to the circuit its electromagnetic inertia. The mechanical counterpart of the term containing L in the *RCL* circuit equation is $m\ddot{x}$ proportional to the acceleration.

To resume, the *RCL* circuit equation for an electric circuit fed with a voltage $V(t)$ is

$$L\ddot{I}(t) + R\dot{I}(t) + \frac{1}{C} I(t) = \dot{V}(t) \quad (= f.e.m.),$$

and it looks like the equation $m\ddot{x}(t) + f\dot{x}(t) + \kappa x(t) = F(t)$ of a material point of mass m attached to a spring and moving in the x-direction with a fluid-type friction pulled out from its rest position by a force $F(t)$. In both cases, $V(t)$ and $F(t)$ can be sinusoidal functions of the time t.

7.2. An example of application

We consider the test case of Fig. 7. A linear material characterized by constant values of σ, ϵ, μ, fills in the space between two perfectly conducting plane electrodes. The two plates are connected on the one side to a sinusoidal voltage generator with frequency ω ($\tau = 2\pi/\omega$) and to a one-port element on the other side. We neglect the distortion of the fields at the wedges of the plates and work, for simplicity, in the frequency domain assuming for example that $\mathbf{E}(z,t) = E(z)e^{i\omega t}\mathbf{i}_x$ where $E(z)$ is the amplitude of the electric field depending only on the z-coordinate. In the configuration of Fig. 7, we have: $E, D, J \parallel \mathbf{i}_x$ and $H, B \parallel \mathbf{i}_y$.

7.2.1. The full model

The equations for the amplitude of the electromagnetic field are

$$\frac{dE}{dz} = -i\omega B, \qquad -\frac{dH}{dz} = J + i\omega D, \tag{23}$$

with constitutive relations $D = \epsilon E$, $B = \mu H$, $J = \sigma E$, and boundary conditions

$$E(-\ell) = -V_g/d, \qquad H(-\ell) = I_g/w,$$
$$E(0) = -V_e/d, \qquad H(0) = I_e/w,$$

where V_e, I_e are respectively the voltage and the current at the end of the plates ($z = 0$), V_g, I_g the voltage and the current at the generator ($z = -\ell$). Assuming that V_g is known and that $z = 0$ an open circuit is connected, $I_e = 0$, only V_e and I_g have to be determined. Due to the problem configuration, a voltage between the plates and a current flowing into the plates can be defined for each value of z as follows:

$$V(z) = -dE(z), \qquad I(z) = H(z)w.$$

Deriving V w.r.t. z and using the first equation of (23), we have

$$\frac{dV}{dz} = i\omega LI, \qquad L = \frac{\mu d}{w}, \qquad V(-\ell) = V_g.$$

Deriving I w.r.t. z and using the second equation of (23), we get

$$\frac{dI}{dz} = GV + i\omega CV, \qquad G = \frac{w\sigma}{d}, \qquad C = \frac{w\epsilon}{d}, \qquad I(0) = 0.$$

The quantities L, G, C are, respectively, the inductance, the conductance ($G = 1/R$) and the capacitance per unit length of the system in Fig. 7. With second derivation of V w.r.t. z we obtain

$$\frac{d^2V}{dz^2} - i\omega L(G + i\omega C)V = 0, \tag{24}$$

where the complex constant

$$\gamma = \sqrt{i\omega L(G + i\omega C)} = z_1 + iz_2$$

is the propagation constant. The real part of the propagation constant (z_1) is defined as the attenuation constant while the imaginary part (z_2) is defined as the phase constant. The attenuation constant defines the rate at which the fields of the wave are attenuated as the wave propagates. An electromagnetic wave propagates in an ideal (lossless) media without attenuation ($z_1 = 0$). The phase constant defines the rate at which the phase changes as the wave propagates. In particular, using the definition of G, L, C, we get

$$\gamma = \sqrt{-(\omega^2\mu\epsilon - i\omega\sigma\mu)} = i\omega\sqrt{\mu\epsilon}\sqrt{1 - i\frac{1}{\omega\tau_e}} = i\tilde{\gamma}.$$

The solution of (24) has the form

$$V = c_1 e^{i\tilde{\gamma}z} + c_2 e^{-i\tilde{\gamma}z}, \tag{25}$$

with the two (real) constants c_1 and c_2 to be determined by imposing the conditions $I(0) = 0$ and $V(-\ell) = V_g$. We get $c_1 = c_2 = V_g/(e^{-i\tilde{\gamma}\ell} + e^{i\tilde{\gamma}\ell})$. For the current we have

$$I = \frac{c_1}{Z}\left(e^{-i\tilde{\gamma}z} - e^{i\tilde{\gamma}z}\right), \qquad Z = \frac{\omega L}{\tilde{\gamma}}. \tag{26}$$

7.2.2. Case (i), the RC model

The equations to be solved are the EQS ones, which for the considered example read

$$\frac{dE}{dz} = 0, \qquad -\frac{dH}{dz} = J + i\omega D, \tag{27}$$

whose solution is

$$E(z) = \tilde{E}, \qquad H(z) = -z(\sigma + i\omega\epsilon)\tilde{E} \tag{28}$$

where $\tilde{E}$ is a constant. Since E is constant, $V_g = V_e$. For the current, we get $I_g = H(-\ell)w = \ell w(\sigma + i\omega\epsilon)\frac{V_g}{d}$. The admittance $\mathcal{Y}$ of the equivalent circuit is

$$\mathcal{Y} = \frac{I}{V} = \frac{\ell w\sigma}{d} + i\omega\frac{\ell w\epsilon}{d} = \ell G + i\omega\ell C.$$

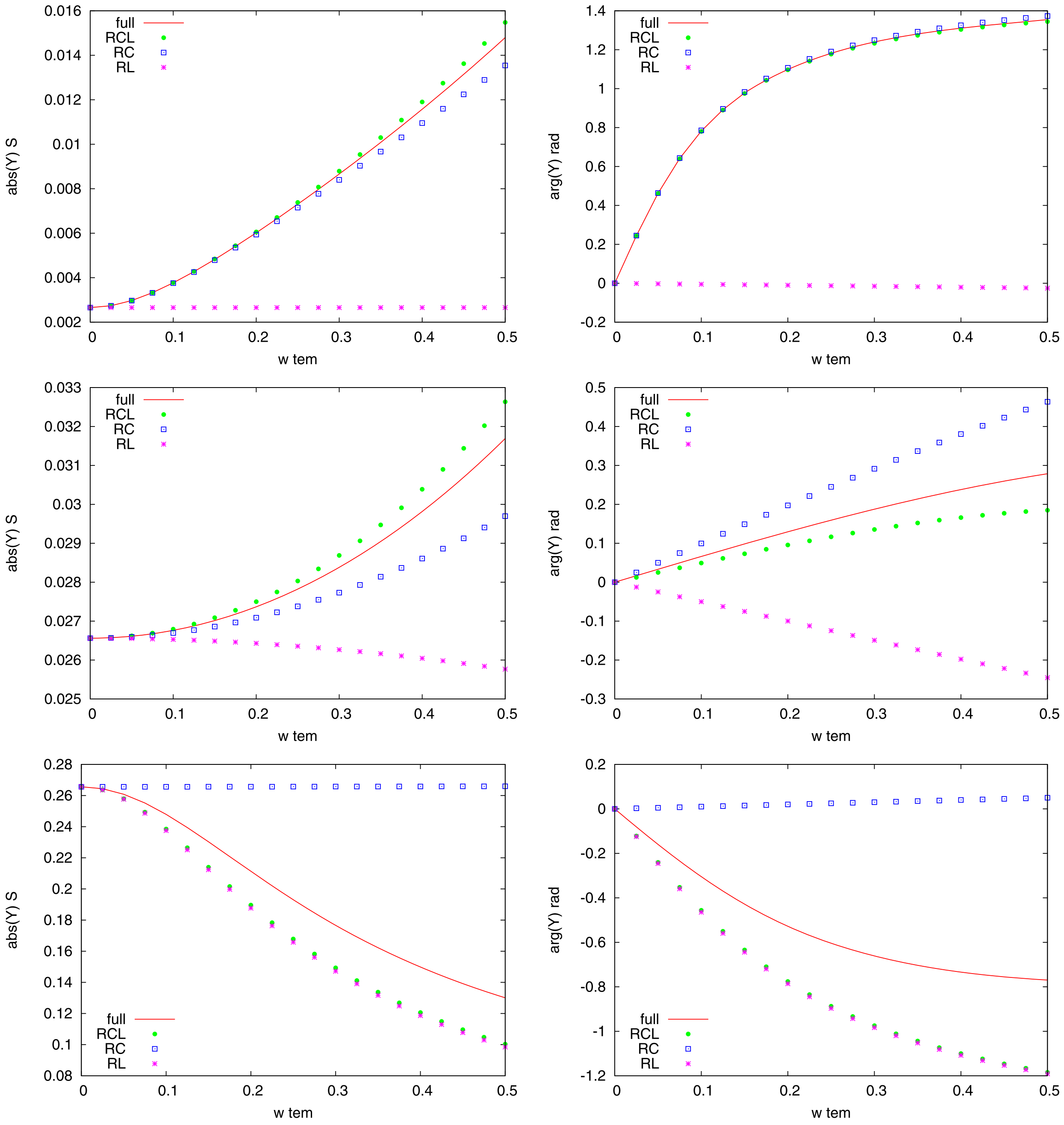

Fig. 8. Behavior of the modulus (left column, in Siemens) and argument (right column, in radiant) of the complex number $\mathcal{Y} = \frac{I}{V}$ which represents the frequency response of the equivalent circuit, compared with the true (full) response characterized $\mathcal{Y}$ with V and I given in (25) and (26) respectively. Case (i) is labeled *RC*, case (ii) *RL*, case (iii) *RCL* (the label *tem* stands for τ_{em}). Top figures correspond to $\tau_e/\tau_{em} = 0.10$, middle figures to $\tau_e/\tau_{em} = 1$, and bottom ones to $\tau_e/\tau_{em} = 10$.

7.2.3. Case (ii), the RL model

The equations to be solved are the MQS ones, which for the considered example are

$$\frac{dE}{dz} = -i\omega B, \qquad -\frac{dH}{dz} = J, \tag{29}$$

which combined with the constitutive relations yield

$$\frac{d^2H}{dz} - i\omega\mu\sigma H = 0.$$

We now look for $\gamma = z_1 + iz_2$ such that $i\omega\mu\sigma = \gamma^2$. By identifying the real and imaginary of the left- and right-hand sides, we get

$$\gamma = \frac{(1+i)}{\delta}, \qquad \delta = \sqrt{\frac{2}{\omega\mu\sigma}},$$

where δ is the well-known penetration depth of the magnetic field in the conductor. The solution is

$$H(z) = c_1 e^{-\gamma z} + c_2 e^{\gamma z}, \qquad I(z) = wH(z), \tag{30}$$

and

$$E(z) = \frac{\gamma}{\sigma}\left(c_1 e^{-\gamma z} - c_2 e^{\gamma z}\right), \qquad V(z) = -dE(z). \tag{31}$$

The two (complex) constants c_1 and c_2 have to be determined by imposing the boundary conditions $I(-\ell) = I_g$ and $I(0) = I_e = 0$. We thus obtain $c_1 = -c_2$ and $c_1 = (I_g/w)(e^{\gamma\ell} - e^{-\gamma\ell})^{-1}$. As a result we get

$$I(z) = I_g \frac{e^{-\gamma z} - e^{\gamma z}}{e^{\gamma\ell} - e^{-\gamma\ell}}, \qquad V(z) = I_g\left(\frac{\gamma d}{\sigma w}\right)\frac{(e^{-\gamma z} + e^{\gamma z})}{(e^{-\gamma\ell} - e^{\gamma\ell})}.$$

Recalling that the following Taylor expansions $e^x + e^{-x} = 2 + x^2 + O(x^4)$ and $e^x - e^{-x} = 2x + O(x^3)$ hold for $|x| \ll 1$, the impedance $\mathcal{Z} = 1/\mathcal{Y}$ of the equivalent circuit is (for $|\gamma\ell| \ll 1$)

$$\mathcal{Z} = \frac{V}{I} = \frac{\gamma d}{w\sigma}\frac{(2+\gamma^2\ell^2)}{2\gamma\ell} = \frac{d}{w\sigma\ell} + \frac{d\ell\gamma^2}{2w\sigma} = \frac{d}{w\sigma\ell} + i\omega\frac{\mu d\ell}{2w} = \frac{1}{G\ell} + i\omega\frac{L}{2}\ell.$$

7.2.4. Case (iii), the RCL model

The equations to be solved are (23) which have been solved at the beginning of the section. We are in the case where $\tau_{em} \ll \tau$ which implies $|\beta\ell| \ll 1$. By applying Euler relations ($e^{\pm ix} = \cos(x) \pm i\sin(x)$) and Taylor expansions $2\cos(x) = 2 - x^2 + O(x^4)$ and $2\sin(x) = 2x + O(x^3)$ when $|x| \ll 1$, from (25), (26) we get

$$\mathcal{Z} = \frac{i\omega L}{2\tilde{\gamma}^2\ell}\left(2 - \tilde{\gamma}^2\ell^2\right).$$

8. Numerical results and conclusions

The solution of the full model is compared with that of reduced models for different time-ranges. The parameters used in the sequel for the numerical simulations are: $d = 0.01$ m, $w = 0.1$ m, $\ell = 1$ m, $\epsilon = \epsilon_0$, $\mu = \mu_0$. The conductivity σ has been varied in order to get different ratios for τ_{em}/τ_e. From the results in Fig. 8 we can observe that when $\tau_e/\tau_{em} \ll 1$ the model *RC* provides the correct answer, when $\tau_e/\tau_{em} \gg 1$ the model *RL* is the more correct one, but neither the *RC* nor the *RL* are correct when $\tau_e/\tau_{em} \approx 1$. The *RCL* model reproduces correctly the results of the full model (in the case $\tau_{em}/\tau \ll 1$) even when $\tau_e/\tau_{em} \approx 1$, as expected. To make more valid the approximation used to get the equivalent circuit, one could split the original system into a number of smaller pieces. A cascade of equivalent circuits connected in parallel would give numerical results which are closer to the ones of the full model.

The focus of this publication is slowly time-varying fields, which can be approximated in the majority of cases by EQS or MQS fields. The EQS approximation is usually applied to field problems arising from high-voltage technology or micro-electronics, where capacitive and resistive effects have to be taken into account whereas inductive effects can be neglected. On the contrary, if inductive and resistive effects have to be considered, the MQS approximation applies. Instances for this include the design analysis of electrical machines or loss computations in transformers at power frequencies. In this approximation, capacitive effects are neglected. The applicability of quasi-static approximations for slowly time-varying electromagnetic fields was investigated using only the characteristic quantities resulting from a dimensional analysis of Maxwell's equations. The role of the "small parameters" involved in the analysis has been underlined and the validity of the models numerically tested in a simple case inspired from [13]. Some open questions remain, such as a mathematical justification of the EQS model, a physical justification of the Darwin one, that we probably try to address in future work.

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