On some applications of Galilean electrodynamics of moving bodies

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We discuss the seminal article by Le Bellac and Lévy-Leblond in which they identified two Galilean limits (called "electric" and "magnetic" limits) of electromagnetism and their implications. Recent work has shed new light on the choice of gauge conditions in classical electromagnetism. We show that the recourse to potentials is compelling in order to demonstrate the existence of both (electric and magnetic) limits. We revisit some nonrelativistic systems and related experiments, in the light of these limits, in quantum mechanics, superconductivity, and the electrodynamics of continuous media. Much of the current technology where waves are not taken into account can be described in a coherent fashion by the two limits of Galilean electromagnetism instead of an inconsistent mixture of these limits.
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I. INTRODUCTION

The purpose of this article is to emphasize the relevance of Galilean electromagnetism, recognized in 1973 by Le Bellac and Lévy-Leblond.[1] They observed that there exist not only one, but two well-defined Galilean limits of electromagnetism: the magnetic and electric limits.[1] "Galilean" means that the theory satisfies the principle of relativity in its Galilean form (also referred to by the somewhat misleading term "nonrelativistic").

We point out that some physical phenomena, often explained with special relativity, can also be explained by appropriately defined Galilean limits. In other words, such phenomena could have been understood without recourse to special relativity had the Galilean limits of electrodynamics been correctly defined. Our purpose is not to argue that these limits are alternatives to Lorentz-covariant electrodynamics in the relativistic context, but that some nonrelativistic phenomena are described erroneously by relativistic electrodynamics (or by nonrelativistic limits not compatible with Galilean covariance).

Our general goal is to show that we must be careful when investigating nonrelativistic limits, and that well-defined Galilean-covariant theories allow us to describe more nonrelativistic phenomena than is usually believed. The latter point means that some concepts, which are thought to be purely relativistic, can actually be understood within the realm of Galilean physics. An example is given by the concept of spin.[2] Note that when a set of equations is said to be covariant with respect to specific transformations of space-time, the form of the equations remains the same. Covariance does not mean invariance of the variables. However, invariance always implies covariance.

We have summarized and discussed various approaches to Galilean electromagnetism in a recent article,[3] but we have learned since then of the existence of several studies emphasizing the applications of quasistatic regimes in microelectronics,[4] biosystems engineering, medical engineering, electromagnetic computations,[5] and teaching.[6] This has motivated us to expand our earlier investigation along the lines described in the following paragraph.

In Sec. II we illustrate some applications of the Galilean electrodynamics of moving bodies. We then re-examine the gauge conditions and their compatibility with Lorentz and Galilean covariance, and emphasize that the use of potentials is necessary to obtain both Galilean limits, which were only stated in Ref. 1. Then we comment briefly on the connection between the two limits and the Faraday tensor (and its dual). In Sec. V we discuss Feynman's proof of the magnetic limit of the Maxwell equations. Section VI contains a few comments about superconductivity interpreted as analogous to the Galilean magnetic limit of electromagnetism and discusses gauge potentials. In Secs. VII and VIII we question the current understanding of the electrodynamics of moving bodies by examining the Trouton–Noble experiment in a Galilean context as well as the introductory example used by Einstein in his famous work on special relativity. We conclude with some comments on the intrinsic use by Maxwell of both limits, one century before Ref. 1.

II. REVIEW OF GALILEAN ELECTROMAGNETISM

We first briefly review Galilean electromagnetism and set up the main equations that we use later. A Lorentz transformation acts on space-time coordinates as follows (see, for instance, see Ref. 7, Sec. 7.2):

$$\mathbf{x}' = \mathbf{x} - \gamma \mathbf{v} t + (\gamma - 1)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{x})}{\mathbf{v}^2}, \tag{1a}$$

$$t' = \gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right), \tag{1b}$$

where $\mathbf{v}$ is the relative velocity, and $\gamma = 1/\sqrt{1-(v/c)^2}$. When $v \ll c$, Eq. (1) reduces to the Galilean transformation of space-time:

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \tag{2a}$$

$$t' = -t. \tag{2b}$$

Because Galilean kinematics involves the time-like condition

$$c\Delta t \gg \Delta x, \quad (3)$$

there is no other possible limit than the one given in Eq. (2). As we shall see, this limit process is not the same for transformations of electric and magnetic fields.

Under the Lorentz transformation in Eq. (1), the electric and magnetic fields in vacuum transform as

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{\mathbf{v}^2}, \quad (4a)$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) + (1-\gamma)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{B})}{\mathbf{v}^2}. \quad (4b)$$

If we take the limit $v/c \to 0$, we find

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad (5a)$$

$$\mathbf{B}' = \mathbf{B}. \quad (5b)$$

As we will see shortly, these transformations form a legitimate limit called the *magnetic limit* of electromagnetism. We might be tempted to consider the limit $\gamma \to 1$, which leads to the following incorrect transformations:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad (6a)$$

$$\mathbf{B}' = \mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}, \quad (6b)$$

which is not a valid transformation. In particular, it does not satisfy the group composition law.[1] That is, if we consider a third reference frame and express E'' and B'' in terms of E' and B', then the ensuing relations between E'' and B'' in terms of E and B do not have the form given in Eq. (6).

However, Eq. (4) allows us to obtain, in addition to Eq. (5), another perfectly well-defined Galilean limit. To do so, we must compare the moduli of the electric field E and the magnetic field cB, in analogy with Eq. (3). For large magnetic fields, Eq. (4) reduces to the magnetic limit of electromagnetism, already given in Eq. (5):

$$\mathbf{E}'_m = \mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m, \quad (E_m \ll cB_m), \quad (7a)$$

$$\mathbf{B}'_m = \mathbf{B}_m. \quad (7b)$$

The alternative, for which the electric field dominates, leads to the *electric limit*:

$$\mathbf{E}'_e = \mathbf{E}_e, \quad (E_e \gg cB_e), \quad (8a)$$

$$\mathbf{B}'_e = \mathbf{B}_e - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}_e. \quad (8b)$$

The approximations $E_e/c \gg B_e$ and $v \ll c$ together imply that $E_e/v \gg E_e/c \gg B_e$, so that we take $E_e \gg vB_e$ in Eq. (4).

Clearly, the small v/c approximation is subtle and should not be confused with neglecting v/c. Moreover, we must be cautious when referring to "orders of magnitude" or "orders of approximation" of v/c because, from a group-theoretical viewpoint, physical theories are either exactly relativistic or Galilean covariant. But they cannot be Galilean covariant to some specific order. To illustrate this point, let us consider the Taylor expansion of the space-time Lorentz transformation, Eq. (1), for a boost along the x axis:

$$x' = \gamma\left(x - \frac{v}{c}ct\right) \approx x - \frac{v}{c}ct + \frac{1}{2}\frac{v^2}{c^2}x - \frac{1}{2}\frac{v^3}{c^3}ct + \cdots, \quad (9a)$$

$$ct' = \gamma\left(ct - \frac{v}{c}x\right) \approx ct - \frac{v}{c}x + \frac{1}{2}\frac{v^2}{c^2}ct - \frac{1}{2}\frac{v^3}{c^3}x + \cdots. \quad (9b)$$

We can readily check that any transformation to some finite order does not satisfy the composition law of group property; that is, the sequence of one boost with velocity v followed by another boost with velocity v' does not have the initial form of the transformations, as may be verified to first order:

$$x' \approx x - \frac{v}{c}ct, \quad (10a)$$

$$ct' \approx ct - \frac{v}{c}x. \quad (10b)$$

Equation (10) is unlike the Galilean transformation, Eq. (2), which does satisfy the composition property of group transformations. (Note that the composition property is also satisfied by Carroll kinematics,[8]

$$x' \approx x, \quad (11a)$$

$$ct' \approx ct - \frac{v}{c}x, \quad (11b)$$

which we will not discuss further, because it entails some contradiction with the hypothesis of causality.)

Because we will emphasize the use of scalar and vector potentials $(V, \mathbf{A})$, let us consider their transformation properties. Under a Lorentz transformation, Eq. (1), they become

$$\mathbf{A}' = \mathbf{A} - \frac{\gamma \mathbf{v} V}{c^2} + (\gamma - 1)\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{A})}{\mathbf{v}^2}, \quad (12a)$$

$$V' = \gamma(V - \mathbf{v}\cdot\mathbf{A}). \quad (12b)$$

When $v \ll c$ and $A \ll cV$, Eq. (12) reduces to the electric limit of potential transformations:

$$\mathbf{A}'_e = \mathbf{A}_e - \frac{\mathbf{v}V_e}{c^2}, \quad (13a)$$

$$V'_e = V_e. \quad (13b)$$

The electric and magnetic fields are expressed in terms of the potentials as follows:

$$\mathbf{E}_e = -\nabla V_e, \quad \mathbf{B}_e = \nabla \times A_e. \quad (14)$$

Although there exists only one possible condition, Eq. (3), for the space-time manifold, we find here a second limit, obtained by $v \ll c$ and $A \gg cV$, such that Eq. (12) reduces to the magnetic limit of potential transformations:

$$\mathbf{A}'_m = \mathbf{A}_m, \quad (15a)$$

$$V'_m = V_m - \mathbf{v}\cdot\mathbf{A}_m. \quad (15b)$$

In this limit, the electromagnetic field components are given by

$$\mathbf{E}_m = -\nabla V_m - \partial_t \mathbf{A}_m, \quad \mathbf{B}_m = \nabla \times A_m. \quad (16)$$

Finally, let us recall the two Galilean limits of the Maxwell equations. Their relativistic form is written as

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \text{(Faraday)}, \tag{17a}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \text{(Thomson)}, \tag{17b}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\partial_t \mathbf{E}, \quad \text{(Ampère)}, \tag{17c}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \quad \text{(Gauss)}. \tag{17d}$$

The existence of two Galilean limits is not so obvious if we naively take the limit $c \to \infty$. It is stated in Ref. 1 that in the electric limit, the Maxwell equations reduce to

$$\nabla \times \mathbf{E}_e = \mathbf{0}, \tag{18a}$$

$$\nabla \cdot \mathbf{B}_e = 0, \tag{18b}$$

$$\nabla \times \mathbf{B}_e - \frac{1}{c^2}\partial_t \mathbf{E}_e = \mu_0 \mathbf{j}_e, \tag{18c}$$

$$\nabla \cdot \mathbf{E}_e = \frac{1}{\epsilon_0}\rho_e. \tag{18d}$$

Clearly, the main difference with the relativistic Maxwell equations is that here the electric field has zero curl in Faraday's law. In the magnetic limit, the Maxwell equations become

$$\nabla \times \mathbf{E}_m = -\partial_t \mathbf{B}_m, \tag{19a}$$

$$\nabla \cdot \mathbf{B}_m = 0, \tag{19b}$$

$$\nabla \times \mathbf{B}_m = \mu_0 \mathbf{j}_m, \tag{19c}$$

$$\nabla \cdot \mathbf{E}_m = \frac{1}{\epsilon_0}\rho_m. \tag{19d}$$

The displacement current term is absent in Ampère's law.

Throughout this article, we will retain the constants ϵ_0 and μ_0 explicitly. Although they somewhat clutter up the equations, the existence of two nonrelativistic limits can be traced back to the possibility of keeping either one of them finite, while the second one approaches zero. As explained in the conclusion of Ref. 1, we may understand the magnetic limit by keeping μ_0 only, and by writing $\epsilon_0 = 1/\mu_0 c^2$ where c then approaches infinity; for the electric limit, we reverse the roles of μ_0 and ϵ_0.

III. GAUGE CONDITIONS AND GALILEAN ELECTROMAGNETISM

In this section we use the Riemann–Lorenz formulation of classical electromagnetism, which is in terms of scalar and vector potentials instead of fields,[9–12] to describe the two Galilean limits, in contrast to the customary Heaviside–Hertz approach in terms of the electromagnetic fields. Our purpose is to examine some implications for Galilean electromagnetism.

Let us recall how the electric and magnetic limits may be retrieved in this formulation by a careful consideration of orders of magnitude.[3,13] It is natural to define the following dimensionless parameters:

$$\varepsilon \equiv \frac{L}{cT} \quad \text{and} \quad \xi \equiv \frac{j}{c\rho}, \tag{20}$$

where L, T, j, and ρ represent the orders of magnitude of length, time, current density, and charge density, respectively.

The equations of classical electromagnetism in terms of the potentials are[9]

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad \text{(Riemann equations)}, \tag{21a}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}, \tag{21b}$$

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0, \quad \text{(Lorenz equation)}, \tag{21c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v} + q\mathbf{A}) = -q\,\nabla\,(V - \mathbf{v}\cdot\mathbf{A}), \quad \text{(Lorentz force)}. \tag{21d}$$

with $d\mathbf{A}/dt = \partial\mathbf{A}/\partial t + \mathbf{v}\cdot\nabla\mathbf{A}$, where ∇ applies only to $\mathbf{A}(\mathbf{r},t)$ and not to $\mathbf{v} = d\mathbf{r}/dt$.

The quasistatic approximation, $\varepsilon \ll 1$, of Eq. (21) leads to

$$\nabla^2 V \simeq -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla^2 \mathbf{A} \simeq -\mu_0 \mathbf{j}, \tag{22}$$

from which we can define a further dimensionless ratio, $cA/V \simeq j/\rho c$, so that

$$\frac{cA}{V} \simeq \xi. \tag{23}$$

This dimensionless parameter echoes the prescription of Ref. 1 for the fields: In the magnetic limit, the spacelike quantity cA is dominant, whereas in the electric limit, it is the timelike quantity V that dominates.

The definition $\mathbf{E} = -\partial_t \mathbf{A} - \nabla V$ of the electric field takes different forms in the Galilean limits, depending on the order of magnitude of each term, because the Galilean transformations of the potentials differ for the electric and the magnetic limits.[1] Let us evaluate the order of magnitude of the ratio between the two terms:

$$\frac{|\partial_t \mathbf{A}|}{|\nabla V|} \simeq \frac{(A/T)}{(V/L)} \simeq \frac{L}{cT}\frac{cA}{V} \simeq \varepsilon\xi. \tag{24}$$

In the magnetic limit, for which $\xi \gg 1$, Eq. (24) leads to Eq. (16). By calculating the curl, we find $\partial_t \mathbf{B}_m = -\nabla \times \mathbf{E}_m$. Likewise, in the electric limit, for which $\xi \ll 1$, we can neglect $\partial_t \mathbf{A}$, so that we obtain Eq. (14). The curl of this expression leads to $\nabla \times \mathbf{E}_e \simeq \mathbf{0}$.

The choice of gauge conditions allows us to retrieve the two sets of Galilean Maxwell equations in terms of fields, as in Ref. 1. Moreover, as we now show, the gauge conditions are closely related to the nature of the kinematic transformations. In the magnetic limit, the condition $\xi \gg 1$ leads to the Coulomb gauge condition: $\nabla \cdot \mathbf{A}_m = 0$. From the definition of $\mathbf{B}_m$ and the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \tag{25}$$

we see that

$$\nabla \times \mathbf{B}_m = \nabla \times (\nabla \times \mathbf{A}_m) = \nabla(\overbrace{\nabla \cdot \mathbf{A}_m}^{0}) - \nabla^2 \mathbf{A}_m = \mu_0 \mathbf{j}_m. \tag{26}$$

The last term follows from Eq. (22). The displacement current term is missing. The divergence of the electric field in the magnetic limit gives

$$\nabla \cdot \mathbf{E}_m = \nabla \cdot (-\partial_t \mathbf{A}_m - \nabla V_m) = -\partial_t(\overbrace{\nabla \cdot \mathbf{A}_m}^{0}) - \nabla^2 V_m = \frac{\rho_m}{\epsilon_0}, \tag{27}$$

where we have utilized Eq. (22). Equation (27) agrees with Eq. (19d). In the electric limit, the condition $\xi \ll 1$ leads similarly to the Lorenz condition. Proceeding as in the magnetic limit, we begin with the curl of $\mathbf{B}_e$:

$$\nabla \times \mathbf{B}_e = \nabla \times (\nabla \times \mathbf{A}_e) = \nabla(\overbrace{\nabla \cdot \mathbf{A}_e}^{-(\partial_t V_e)/c^2}) - \nabla^2 \mathbf{A}_e = \frac{1}{c^2}\partial_t \mathbf{E}_e + \mu_0 \mathbf{j}_e. \tag{28}$$

From the divergence of $\mathbf{E}_e$, we find

$$\nabla \cdot \mathbf{E}_e = \nabla \cdot (-\nabla V_e) = -\nabla^2 V_e = \frac{\rho_e}{\epsilon_0}, \tag{29}$$

where we have used Eq. (22).

To summarize the preceding discussion, the choice of a gauge condition is dictated by the relativistic versus Galilean nature of the problem. Moreover, we claim that recourse to the potentials is necessary to demonstrate mathematically the existence of the Galilean limits which was only stated in Ref. 1. The role of potentials and gauge conditions in quasistatic regimes was pointed out only recently by Dirks[4] and Larsson,[6] although the problem was not handled correctly, as we have done with the Riemann–Lorenz formulation. Indeed, both Refs. 4 and 6 use the (erroneous) equation:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} + \frac{1}{c^2}\nabla \frac{\partial V}{\partial t}, \tag{30}$$

obtained by using the (magnetic Galilean covariant: $\xi \gg 1$) Coulomb gauge condition together with the full set of (Lorentz covariant: $\xi \simeq 1$) Maxwell equations in terms of the fields without remarking that the temporal terms are negligible compared to the spatial terms because $\epsilon \ll 1$.

The Lorenz gauge condition is compatible in the relativistic context as well as the electric Galilean limit. However, the Coulomb gauge condition is only compatible with the Galilean magnetic limit because it is not covariant with respect to either the Lorentz transformations or the Galilean electric transformations. We refer the interested reader to a discussion of the physical meaning that can be ascribed to the various gauge conditions.[14]

Galilean electromagnetism sheds a new light on the prerelativity era. A careful reading of Maxwell's famous *Treatise on Electricity and Magnetism* reveals that he was actually working with the electric limit in the discussion of dielectric materials (see Chaps. II to V).[15] Likewise, in his treatment of ohmic conductors and induced magnetic fields in Vol. II, the magnetic limit was employed implicitly, except in the chapters on the theory of light propagation, where he introduced by hand the displacement current term into the magnetic limit equations to demonstrate that light is a transverse electromagnetic wave.[15] But, as we have seen in the special case of the electric limit (and in general in relativity), the displacement current follows from choosing the Lorenz gauge, and Maxwell (wrongfully) kept the Coulomb gauge within the relativistic context for the fields. (For more details, see Ref. 16.) This problem prompted Hertz and Heaviside to relinquish potentials and instead cast the Maxwell equations in terms of fields. Following Hertz's approach, Einstein subsequently expressed the Maxwell equations in terms of fields (that is, in the Heaviside–Hertz formulation), whereas Poincaré wrote the Maxwell equations in terms of the potentials (that is, the Riemann–Lorenz formulation) by adopting the Lorenz condition in a relativistic context.[16]

IV. THE FARADAY TENSOR AND ITS DUAL

In special relativity it is well known that the Faraday tensor

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (\mu,\nu = 0,1,2,3), \tag{31}$$

and its dual

$$^{*}F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{32}$$

have the same physical meaning. This equivalence is not valid in Galilean electromagnetism. As pointed out in Ref. 17 and recently discussed by Rynasiewicz,[18] the Galilean transformations of the Faraday tensor and its dual tensor lead to the electric or the magnetic limit, respectively.[17] The effect of the duality operation amounts to exchanging **E** and **B**:

$$\mathbf{E} \to c\mathbf{B} \quad \text{and} \quad \mathbf{B} \to -\mathbf{E}/c. \tag{33}$$

We recover the magnetic and electric limits, Eqs. (7) and (8), by applying the duality transformations directly to the electric transformations of the fields to obtain the magnetic transformations and vice versa.

Earman also noted[17] that the field transformations of the magnetic limit are obtained when **E** and **B** are expressed in terms of covariant or $\binom{0}{2}$ tensor $F_{\mu\nu}$, whereas the electric limit is obtained when the fields transformations are calculated by using the contravariant or $\binom{2}{0}$ tensor $F^{\mu\nu}$. Let us illustrate it briefly with

$$A^\mu = \left(\frac{V}{c}, \mathbf{A}\right), \quad A_\mu = \left(\frac{V}{c}, -\mathbf{A}\right), \tag{34}$$

as well as

$$\partial^\mu = \left(\frac{1}{c}\partial_t, -\nabla\right), \quad \partial_\mu = \left(\frac{1}{c}\partial_t, \nabla\right). \tag{35}$$

The magnetic limit follows from the relation

$$F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}, \tag{36}$$

where the Galilean transformation matrix $\Lambda_\mu{}^\nu$ is defined by the four-gradient transformation, $\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$, so that

$$\Lambda_\mu{}^\nu = \begin{pmatrix} 1 & \frac{v_x}{c} & \frac{v_y}{c} & \frac{v_z}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (37)$$

The index μ denotes the line of each entry. We find, for example,

$$\frac{E'_x}{c} = F'_{01} = \Lambda_0{}^\mu \Lambda_1{}^\nu F_{\mu\nu}, \qquad (38a)$$

$$= \frac{1}{c}(E_x + v_y B_z - v_z B_y), \qquad (38c)$$

and

$$-B'_z = F'_{12} = \Lambda_1{}^\mu \Lambda_2{}^\nu F_{\mu\nu} = -B_z, \qquad (39)$$

which is Eq. (7).

The electric limit transformations follows from

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}. \qquad (40)$$

The transformation matrix $\Lambda^\mu{}_\nu$ is now defined by the coordinate transformation, $x^\mu = \Lambda^\mu{}_\nu x^\nu$, with $x^\mu = (ct, x, y, z)$, so that

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{v_x}{c} & 1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & 1 & 0 \\ -\frac{v_z}{c} & 0 & 0 & 1 \end{pmatrix}. \qquad (41)$$

For instance, we calculate

$$-\frac{E'_x}{c} = F'^{01} = \Lambda^0{}_0 \Lambda^1{}_\nu F^{0\nu} = -\frac{E_x}{c}, \qquad (42)$$

and

$$B'_z = -F'^{12} = -\Lambda^1{}_\mu \Lambda^2{}_\nu F^{\mu\nu}, \qquad (43a)$$

$$= -\Lambda^1{}_0 F^{02} - \Lambda^2{}_0 F^{10} - F^{12}, \qquad (43b)$$

$$= B_z - \frac{1}{c^2}(v_x E_y - v_y E_x), \qquad (43c)$$

which is Eq. (8).

V. QUANTUM MECHANICS WITH EXTERNAL POTENTIALS

In 1990 Dyson published a demonstration of the Maxwell equations due to Feynman.[19] The demonstration dates back to the 1940s and had remained hitherto unpublished. It was believed to be incomplete because Feynman considered only the homogeneous Maxwell equations, given by Eqs. (17a) and (17b):

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \qquad (44)$$

During the 1990s, some authors revisited this demonstration and noted that the Schrödinger equation admitted external potentials only if they were compatible with the magnetic limit[1] and, therefore, with the Coulomb gauge condition (see Refs. 20 and 21 and the references therein).

From Eq. (19), it is clear that the homogeneous Maxwell equations Eq. (44) are valid only within the magnetic limit because the electric field has zero curl in the electric limit, Eq. (18). This validity condition is a consequence of the Galilean magnetic limit of the four-potential which enters into the Schrödinger equation. Let us recall the statement more precisely (more details can be found in Ref. 20). The Schrödinger equation with external fields $V(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ is written as

$$i\hbar\partial_t \Psi(\mathbf{x},t) = \frac{1}{2m}[-i\hbar\nabla - q\mathbf{A}(\mathbf{x},t)]^2 \Psi(\mathbf{x},t) + V(\mathbf{x},t)\Psi(\mathbf{x},t). \qquad (45)$$

It is covariant under the Galilean transformation, Eq. (2), with

$$\Psi(\mathbf{x},t) \to \Psi'(\mathbf{x}',t') = \text{constant} \times \exp\left[(i/\hbar)\left(-m\mathbf{v}\cdot\mathbf{x} + \frac{1}{2}m\mathbf{v}^2 t + \phi(\mathbf{x},t)\right)\right]\Psi(\mathbf{x},t), \qquad (46a)$$

$$V(\mathbf{x},t) \to V'(\mathbf{x}',t') = V(\mathbf{x},t) - \partial_t\phi(\mathbf{x},t) - \mathbf{v}\cdot(\mathbf{A}(\mathbf{x},t) + \nabla\phi(\mathbf{x},t)), \qquad (46b)$$

$$\mathbf{A}(\mathbf{x},t) \to \mathbf{A}'(\mathbf{x}',t') = \mathbf{A}(\mathbf{x},t) + \nabla\phi(\mathbf{x},t), \qquad (46c)$$

where $\phi(\mathbf{x},t)$ is a scalar function. For $\phi(\mathbf{x},t)=0$, which corresponds to pure Galilean boosts, Eq. (46) reduces to the magnetic limit of Galilean transformations of the potentials, Eq. (15). Hence, we can say that Galilean covariance selects the gauge.

In a subsequent study, Holland and Brown have shown that the Maxwell equations admit only an electric limit provided that the source is a Dirac current.[22] In addition, they showed that the Dirac equation admits both Galilean limits, just like the Maxwell equations, consistent with earlier results by Lévy-Leblond.[2] To summarize, what Feynman did not (actually, could not) realize is that he had derived only the part of the Maxwell equations compatible with the Galilean covariant magnetic limit, that is, the homogeneous equations.

VI. SUPERCONDUCTIVITY

Superconductivity also enters into the realm of the magnetic limit; it selects the Coulomb gauge condition as a necessary consequence of Galilean covariance. As en example, consider the London equation, which states that the current density is proportional to the vector potential:[23]

$$\mathbf{p} = m^*\mathbf{v} + q^*\mathbf{A} = \mathbf{0}. \qquad (47)$$

The star denotes a quantity describing Cooper pairs.[23] Equation (47) implies that there is a perfect transfer of electromagnetic momentum to kinetic momentum. Hence, contrary to what is usually stated, gauge invariance is not broken by

Table I. Galilean transformations for the excitations, the fields, and the sources for both magnetic and electric limits.

Magnetic Limit	Electric Limit
$\mathbf{B}=\mathbf{B}'$	$\mathbf{E}=\mathbf{E}'$
$\rho=\rho'+\mathbf{v}\times\mathbf{j}'/c^2$	$\rho=\rho'$
$\mathbf{j}=\mathbf{j}'$	$\mathbf{j}=\mathbf{j}'+\rho'\mathbf{v}$
$\mathbf{H}=\mathbf{H}'$	$\mathbf{H}=\mathbf{H}'+\mathbf{v}\times\mathbf{D}'$
$\mathbf{E}=\mathbf{E}'-\mathbf{v}\times\mathbf{B}'$	$\mathbf{D}=\mathbf{D}'$
$\mathbf{M}=\mathbf{M}'$	$\mathbf{P}=\mathbf{P}'$
$\mathbf{P}=\mathbf{P}'+\mathbf{v}\times\mathbf{M}'/c^2$	$\mathbf{M}=\mathbf{M}'-\mathbf{v}\times\mathbf{P}'$

Table II. Boundary conditions compatible with the Galilean transformations for both magnetic and electric limits.

Magnetic Limit	Electric Limit
$\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}$	$\mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1)=0$
$\mathbf{n}\cdot(\mathbf{B}_2-\mathbf{B}_1)=0$	$\mathbf{n}\cdot(\mathbf{D}_2-\mathbf{D}_1)=\sigma$
$\mathbf{n}\cdot(\mathbf{j}_2-\mathbf{j}_1)+\nabla_\Sigma\cdot\mathbf{K}=0$	$\mathbf{n}\cdot(\mathbf{j}_2-\mathbf{j}_1)+\nabla_\Sigma\cdot\mathbf{K}=v_n(\rho_2-\rho_1)-\partial_t\sigma$
$\mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1)=v_n(\mathbf{B}_2-\mathbf{B}_1)$	$\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}+v_n\mathbf{n}\times[\mathbf{n}\times(\mathbf{D}_2-\mathbf{D}_1)]$

superconductivity because the Coulomb gauge condition is implied.[24] Moreover, the Meissner effect can be explained by starting with Ampère's equation written as $\nabla\times\mathbf{B}=\mu_0\mathbf{j}$, that is, without the displacement current term as in the magnetic case, Eq. (19c). Hence, this expression (or more directly $\nabla^2\mathbf{A}\simeq-\mu_0\mathbf{j}$ in the Riemann–Lorenz formulation) together with $\nabla\cdot\mathbf{A}=0$ and the London equation, lead to solutions (in one dimension x) of the type $A\simeq\exp-\lambda x$ (where λ is a constant). Hence, the vector potential (and the magnetic field) only penetrates the superconductor to a depth $1/\lambda$.[23]

We point out that the current density in the magnetic limit (hence in superconductivity) is divergenceless. By taking the divergence of

$$\nabla^2\mathbf{A}\simeq-\mu_0\mathbf{j},\tag{48}$$

and using $\nabla\cdot\mathbf{A}=0$, dictated by Galilean covariance, we end up with $\nabla\cdot\mathbf{j}=0$. The latter does not mean, as is often assumed, that the current is constant in time. Indeed, only the time derivative of the charge density is negligible with respect to the divergence of the current.[3]

As a consequence, superconductivity cannot be associated with a symmetry breaking of gauge invariance but is magnetic Galilean covariant. This unusual statement has been recently advocated by Martin Greiter using a different approach.[24] Note that it is not the whole gauge symmetry, but the global U(1) phase rotation symmetry that is spontaneously violated.

VII. ELECTRODYNAMICS OF CONTINUOUS MEDIA AT LOW VELOCITIES

In 1904 Lorentz claimed that a moving magnet could become electrically polarized.[25] In 1908 Einstein and Laub noted that the Minkowski transformations for the fields and the excitations[26] predict that a moving magnetic dipole induces an electric dipole moment.[27] It is interesting to reexamine these predictions in light of the Galilean electrodynamics of continuous media. If we start from the Minkowski transformations that relate the polarization and the magnetization,[26] we would expect two Galilean limits: one with $\mathbf{M}'=\mathbf{M}$ and $\mathbf{P}'=\mathbf{P}-\mathbf{v}\times\mathbf{M}/c^2$ and the other with $\mathbf{M}'=\mathbf{M}+\mathbf{v}\times\mathbf{P}$ and $\mathbf{P}'=\mathbf{P}$ (Ref. 28, Chap. 9).

In Ref. 3 we derived the fields transformations in Table I. In addition, we display the boundary conditions for moving media in Table II, with $\mathbf{n}$ being the unit vector between two media denoted by 1 and 2, $\mathbf{K}$ the surface current, σ the surface charge, Σ the surface separating both media, and v_n the projection of the relative velocity on the normal of Σ. Therefore, as we presumed, the effects in continuous media predicted by Lorentz and by Einstein and Laub are not purely relativistic because they can be described in a Galilean framework.

VIII. ELECTRODYNAMICS OF MOVING BODIES AT LOW VELOCITIES

Galilean electromagnetism raises doubts about our current understanding of the electrodynamics of moving media. For instance, several experiments (such as the ones by Roentgen,[29] Eichenwald,[30] Wilson,[31] Wilson and Wilson,[32,33] and Trouton and Noble[34–37]) are generally believed to corroborate special relativity. However, as we will show for the Trouton–Noble experiment, there is not always a need for special relativity because the typical relative velocity in these experiments is much smaller than the speed of light. As we have emphasized, the Galilean framework must involve the two limits of electromagnetism. A question that arises is which of the experiments we have mentioned can be explained by either the electric limit, the magnetic limit, or a coherent combination of both.

It is interesting to notice that Carvallo, a notorious antirelativist, used both quasistatic limits as early as 1921 to deny the success of Einstein's theory of relativity.[38] In a sense he was right when he pointed out that the electrodynamics of moving bodies at low velocities could be described in a Galilean-covariant manner by distinguishing the conductors and dielectrics. However, he was wrong to think that the optical properties of moving bodies can be described along the same lines.

The following discussion could be considered to be of historical interest only, but we claim that the electrodynamics of moving media requires the use of the Galilean limits in most laboratory experiments. Because the experimental evidence has been known for a long time, we refer the reader to history[29–37] but with a modern perspective.

A. The Trouton–Noble experiment

Here we explain the Trouton–Noble experiment in a Galilean context corresponding to an experiment involving velocities well below the speed of light. Unlike the relativistic approach, our explanation does not assume the existence of a mechanical torque due to length contraction in order to balance the electromagnetic force. Moreover, the Galilean treatment is sufficient to describe capacitors that are moving at low velocities.

The Trouton–Noble experiment—described in the following—can be thought of as an electromagnetic analogue of the optical Michelson and Morley experiment.[34] It was designed to verify whether it is possible to observe a mechanical velocity of the ether if the luminiferous medium is considered as having parts that can be followed mechanically. Like the Michelson–Morley optical experiment, the Trouton–Noble experiment led to a negative result in the sense that no one was able to detect either an absolute motion with respect to the ether, or a partial entrainment such as in the Fizeau experiment.

In 1905 Einstein suggested that the ether was superfluous, because its mechanical motion was not detected experimentally. Some theorists, such as Poincaré and Lorentz, were reluctant to relinquish the ether as the bearer of the electromagnetic field, despite the fact that they had adopted the relativity principle. In 1920, at a conference in Leyden, Einstein himself referred to the ether as the medium allowing the propagation of gravitational waves, although it cannot be endowed with the characteristics of a material medium.[39] Today, even though the ether is a banished word in modern science, we can use it as did the older Einstein to describe the vacuum with physical (though not mechanical) properties.

Before the advent of special relativity, Hertz, Wien, Abraham, Lorentz, Cohn, and others had used the transformations given in Eq. (6), which is an incoherent mixture of the electric and magnetic Galilean limits. As mentioned, these expressions do not even obey the composition property of group transformations.

The purpose of the Trouton–Noble experiment was to observe the effect of a charged capacitor in motion with an angle θ between the plates and the motion through the ether.[34–37] The electric field in the reference frame of the capacitor generates a magnetic field in the ether frame given by

$$\mathbf{B}' = -\frac{1}{c^2}\mathbf{v} \times \mathbf{E}, \tag{49}$$

where $\mathbf{v}$ is the absolute velocity. Thus we have

$$B' = \frac{1}{c^2} vE \sin\theta. \tag{50}$$

Consequently, there is a localization of magnetic energy density inside a volume $\mathrm{d}V$:

$$\mathrm{d}W = \frac{1}{2}\frac{B'}{\mu_0}\mathrm{d}V = \frac{1}{2}\frac{v^2}{c^2}\epsilon_0 E^2 \sin^2\theta \mathrm{d}V. \tag{51}$$

The volume of the capacitor is $S\ell$ and the total energy between the plates is

$$W = \frac{1}{2}\frac{v^2}{c^2}\epsilon_0 E^2 \sin^2\theta S\ell. \tag{52}$$

If we denote the difference of potential between the plates as $V=E/\ell$, then the capacitor is submitted to the electrical torque

$$\Gamma = -\frac{\mathrm{d}W}{\mathrm{d}\theta} = -\frac{\epsilon_0}{2}\frac{V^2 S}{\ell}\frac{v^2}{c^2}\sin 2\theta, \tag{53}$$

which is a maximum for $\theta=45°$, and zero for $\theta=90°$. Hence, the plates are expected to be perpendicular to the velocity. However, this effect has not been observed experimentally.

To understand the flaw with this argument, first consider the electric limit transformation in Eq. (8). A consequence of this transformation is that the Biot–Savart law follows from the Coulomb law associated with the electric transformation of the magnetic field. In addition, these transformations are compatible only with the approximate set of the Maxwell equations where the time derivative in the Faraday equation vanishes, as in Eq. (18).

We can derive the following electric limit approximation of the Poynting theorem:

$$\partial_t\left(\frac{1}{2}\epsilon_0 E^2\right) + \nabla \cdot \left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right) \simeq -\mathbf{j}\cdot\mathbf{E}. \tag{54}$$

This equation shows that the energy density is of electric origin only. Hence, no electric energy associated with the motional magnetic field can be taken into account within the electric limit, because it is of order $(v/c)^2$ with respect to the static, or quasistatic, electric limit. Thus, the Trouton–Noble experiment does not show any effect in the electric limit. Recall that the electric limit is such that the relative velocity is small compared to the velocity of light c, and the order of magnitude of the electric field is large compared to the product of c and the magnetic field. Of course, special relativity is needed for larger velocities, and we must take into account the additional mechanical torque[41] due to the length variation to explain the negative result (that is, no torque).

In the last paragraph of their 1903 article,[35] Trouton and Noble comment on the source of the negative result being caused by the fact that they considered the energy of the motional magnetic field.[34–37] They suggested that the energy of the magnetic field must have had some origin, and that the electrostatic energy of the capacitor had to decrease by $1/2\epsilon_0 E^2 v^2/c^2$ when it is moving with a velocity v at right angles to its electrostatic lines of force (the electrostatic energy is $1/2\epsilon_0 E^2$). We may assert that the converse situation of a solenoid or magnet in motion will not create a motional magnetic torque because the magnetic energy associated with the motional electric field is negligible compared to the magnetic energy of the static, or quasistatic magnetic field.

B. Einstein's asymmetry

The purpose of this section is to introduce the Einstein asymmetry and to provide, as far as we know, the first explanation exploiting Galilean kinematics instead of relativistic kinematics. In his famous article on the electrodynamics of moving media,[42] Einstein pointed out the importance of whether or not we should ascribe energy to the fields when dealing with motion. In the introduction of Ref. 42 he recalled that Maxwell's electrodynamics, when applied to moving bodies, leads to intrinsic theoretical asymmetries. He illustrated such an asymmetry with the example of the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon depends on the relative motion of the conductor and the magnet, unlike the traditional view advocated by Lorentz in which either one or the other of these bodies is in motion. That is, (1) if the magnet is moving with the conductor at rest, an electric field is induced in the neighborhood of the magnet, producing a current where parts of the conductor are located. (2) If the conductor is in motion and the magnet at rest, then no electric field arises in the neighborhood of the magnet. Lorentz argued that the conductor must contain an electromotive force with no intrinsic energy, but which causes electric currents similar to those produced by the electric forces in case (1), assuming the same relative motion in the two cases. This dual representation of the same phenomena was not acceptable to Einstein.

By applying the Lorentz transformation (obtained in the kinematical analysis of his article) to the Maxwell equations,[43] Einstein replaced Lorentz's explanation by the now famous special relativity explanation, valid for all velocities:

1. (Lorentz) If a unit electric point charge is in motion in an electromagnetic field, there acts on it, in addition to the electric force, an electromotive force, which if we neglect the terms multiplied by the second and higher powers of v/c, is equal to the vector product of the velocity of the charge and the magnetic force, divided by the velocity of light.[42]
2. (Einstein) If a unit electric point charge is in motion in an electromagnetic field, the force acting on it is equal to the electric force that is present at the locality of the charge, and which we determine by transforming the field to a system of coordinates at rest relative to the electrical charge.[42]

Einstein therefore concluded that the analogy is valid with magnetomotive forces, based on the idea that the electromotive force is merely some auxiliary concept due to the fact that the electric and magnetic forces are related to the relative motion of the coordinate system. He then pointed out that the asymmetry mentioned in the introduction of his article now disappears.

The transformations of the electromagnetic field given by the Galilean magnetic limit are sufficient to explain Einstein's thought experiment with the magnet and the conductor, without recourse to Lorentz covariance.[44] The magnetic Poynting theorem can explain why an energy cannot be ascribed to the motional electric field in Einstein's thought experiment,

$$\partial_t\left(\frac{B^2}{2\mu_0}\right) + \nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right) \simeq -\mathbf{j}\cdot\mathbf{E}, \tag{55}$$

which is the magnetic analogue of Eq. (54).

Hence, the second postulate (invariance of the velocity of light) used by Einstein is not required to explain the thought experiment. The relativity principle and the magnetic Galilean transformations are sufficient, together with the fact that the relative velocity involved in such an experiment is much smaller than the velocity of light. Hence, in the low-velocity regime, we propose the following explanation of Einstein's asymmetry:

3. If a unit electric point charge is in motion in an electromagnetic field, the force acting on it is equal to the electric force that is present at the locality of the charge. We determine the force by a Galilean magnetic transformation of the field to a system of coordinates at rest relative to the electrical charge.

Einstein was correct in replacing Lorentz's explanation because Lorentz thought that the vector product of the velocity with the magnetic field was not an electric field (which is why Lorentz called it the electromotive field). But, because Ref. 1 was not yet available, Einstein did not notice that the same vector product was an effective electric field due to a transformation of the Galilean magnetic limit. Pauli also offered a solution to the asymmetry problem in his textbook on electrodynamics, but he assumed that his calculations were only a first order approximation of the relativistic demonstration.[45] He did not acknowledge the existence of the Galilean magnetic limit.

To summarize, Einstein's explanation to remove the asymmetry is completely valid. However, special relativity is not necessary to remove it, but only sufficient. It is ironic that the thought experiment that led Einstein to special relativity could have been explained by Galilean relativity if the magnetic limit had been known by him at that time.

As pointed out by Keswani and Kilminster,[46] Maxwell resolved Einstein's asymmetry within the formalism of the magnetic limit when he stated that for all phenomena related to closed circuits and the current within them, whether the coordinate system be at rest or not, is immaterial. In Article 600 of his treatise,[15] Maxwell explained that the formula for the electromotive intensity (in its modern sense and not in the sense of Lorentz) is of the same type, whether the motion of the conductors refers to fixed axes or to moving axes, because the only difference is that for moving axes the electric potential V becomes $V' = V - \mathbf{v}\cdot\mathbf{A}$. Maxwell claimed that whenever a current is produced within a circuit C, the electromotive force is equal to $\int_C \mathbf{E}'\cdot d\mathbf{s}$, and the value of V therefore disappears from this integral, so that the term $-\mathbf{v}\cdot\mathbf{A}$ has no influence on its value.[15] In a not well-known paper,[47] Maxwell clearly formulated the principle of relativity within the Galilean magnetic limit context: The currents in any system are the same, whether the conducting system or the inducing system are in motion, provided the relative motion is the same.

IX. CONCLUDING REMARKS

One century after the relativity revolution occurred, and more than thirty years after the work of Lévy-Leblond and Le Bellac,[1] Galilean electromagnetism is becoming a field of current research, because it allows physicists and engineers to explain low-energy experiments involving the electrodynamics of moving media without the formalism of special relativity.

We have re-examined gauge conditions in connection with Lorentz and Galilean covariance. After a brief comment on the two Galilean limits of electromagnetism and the Faraday tensor, we have recalled the importance of the magnetic limit in Feynman's proof of the Maxwell equations as well as in superconductivity. Finally, we have questioned our current understanding of the electrodynamics of moving bodies by examining the Trouton–Noble experiment and the example used by Einstein in his famous article on special relativity.

For low velocities it is clear that the effects of special relativity, such as length contraction, cannot explain (as it was believed) the corresponding experiments because these effects are negligible. In the realm of mechanics, we might ask what would have happened if Newton had been born after Einstein. The situation is somewhat analogous for electromagnetism.

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